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Measure and Integration Solutions 16

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra. Define σ on $\mathcal{B}(\mathbb{R})$ by $\sigma(\Gamma) = \displaystyle\sum_{n \in \mathbb{Z} \setminus \{0\} \cap \Gamma} \frac{1}{n^2}$.

   (a) Show that σ is a measure on $\mathcal{B}(\mathbb{R})$ such that $\sigma \perp \lambda$, where λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

   (b) Let $f \in L^1(\lambda)$ be non-negative, and define μ on $\mathcal{B}(\mathbb{R})$ by $\mu(\Gamma) = \int_\Gamma f \, d\lambda$. Let $\nu = \mu + \sigma$. Find the Lebesgue decomposition of ν with respect to λ.

   Proof(a) It is easy to check that σ is σ-additive, and that $\lambda(\mathbb{Z}) = \sigma(\mathbb{Z}^c) = 0$. Thus, $\sigma \perp \lambda$.

   Proof(b) Since $\mu \ll \lambda$ and $\sigma \perp \lambda$, then by the uniqueness of the Lebesgue decomposition, we have that $\mu = \nu_a$ and $\sigma = \nu_\sigma$.

2. Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every measurable function $F : E \to \mathbb{R}$ one has

   $$\int_E F \, d\mu = \int_E F h \, d\nu$$

   in the sense that if one integral exists, then the other integral also exists, and they are equal.

   Proof Suppose first that $F = 1_A$ is the indicator function of some measurable set $A \in \mathcal{B}$. Then,

   $$\int_E F \, d\mu = \mu(A) = \int_A h \, d\nu = \int_E 1_A h d\nu = \int_E F h d\nu.$$

   Suppose now that $F = \displaystyle\sum_{k=1}^n \alpha_k 1_{A_k}$ is a non-negative measurable step function. Then,

   $$\int_E F \, d\mu = \sum_{k=1}^n \alpha_k \mu(A_k) = \sum_{k=1}^n \alpha_k \int_E 1_A h d\nu = \int_E \sum_{k=1}^n \alpha_k 1_A h d\nu = \int_E F h d\nu.$$

   Suppose that F is a non-negative measurable function, then there exists a sequence of non-negative measurable step functions F_n such that $F_n \uparrow F$. Then, $F_n h \uparrow F h$, and by the Monotone Convergence Theorem,

   $$\int_E F \, d\mu = \lim_{n \to \infty} \int_E F_n \, d\mu = \lim_{n \to \infty} \int_E F_n h d\nu = \int_E F h d\nu.$$

Finally, suppose that F is a measurable function. Then,

$$\int_E F^+ \, d\mu = \int_E F^+ h \, d\nu \text{ and } \int_E F^- \, d\mu = \int_E F^- h \, d\nu.$$

From the above we see that $\int_E F \, d\mu$ exists if and only if $\int_E Fh \, d\nu$ exists. If both integrals exist, then

$$\int_E F \, d\mu = \int_E F^+ \, d\mu - \int_E F^- \, d\mu = \int_E F^+ h \, d\nu - \int_E F^- h \, d\nu = \int_E Fh \, d\nu.$$

3. Suppose that μ_i, ν_i are finite measures on $(E, \mathcal{B})$ with $\mu_i \ll \nu_i$, $i = 1, 2$. Let $\nu = \nu_1 \times \nu_2$ and $\mu = \mu_1 \times \mu_2$.

   (a) Show that $\mu \ll \nu$.

   (b) Prove that $\frac{d\mu}{d\nu}(x, y) = \frac{d\mu_1}{d\nu_1}(x) \cdot \frac{d\mu_2}{d\nu_2}(y)$ ν a.e.

   Proof(a) Let $\Gamma \in \mathcal{B} \times \mathcal{B}$ be such that $\nu(\Gamma) = 0$. By problem 3, exercises 13 it follows that $\nu_1(\Gamma(x_2)) = 0$ ν_2 a.e. and $\nu_2(\Gamma(x_1)) = 0$ ν_1 a.e. Since $\mu_i \ll \nu_i$, $i = 1, 2$, then $\mu_1(\Gamma(x_2)) = 0$ μ_2 a.e. and $\mu_2(\Gamma(x_1)) = 0$ μ_1 a.e. Thus, by problem 3, exercises 13 it follows that $\mu(\Gamma) = 0$. Therefore, $\mu \ll \nu$.

   Proof(b) First recall that $\frac{d\mu}{d\nu}$ is the unique ν a.e. $L^1(\mu)$ function satisfying $\mu(A) = \int_A \frac{d\mu}{d\nu}$. Consider for any $A \in \mathcal{B} \times \mathcal{B}$, by Tonelli's theorem and problem 3 above, we have

$$\begin{aligned}
\int_A \frac{d\mu_1}{d\nu_1}(x) \cdot \frac{d\mu_2}{d\nu_2}(y) d\nu(x, y) &= \int_E \int_E 1_A(x, y) \frac{d\mu_1}{d\nu_1}(x) \cdot \frac{d\mu_2}{d\nu_2}(y) d\nu_2(y) d\nu_1(x) \\
&= \int_E \int_E 1_A(x, y) \frac{d\mu_1}{d\nu_1}(x) d\mu_2(y) d\nu_1(x) \\
&= \int_E (\int_E 1_A(x, y) d\mu_2(y)) \frac{d\mu_1}{d\nu_1}(x) d\nu_1(x) \\
&= \int_E \int_E 1_A(x, y) d\mu_2(y) d\mu_1(x) \\
&= (\mu_1 \times \mu_2)(A) = \mu(A) = \int_A \frac{d\mu}{d\nu}(x, y) d\nu(x, y).
\end{aligned}$$

   Since this is true for all $A \in \mathcal{B} \times \mathcal{B}$, it follows from problem 3, exercises 9 that $\frac{d\mu}{d\nu}(x, y) = \frac{d\mu_1}{d\nu_1}(x) \cdot \frac{d\mu_2}{d\nu_2}(y)$ ν a.e.

4. Let $(E, \mathcal{B})$ be a measurable space, μ a finite measures on $(E, \mathcal{B})$ and ν a σ-finite measure on $(E, \mathcal{B})$. Show that $\mu \ll \nu$ **if and only if** for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{B}$ with $\nu(A) < \delta$, then $\mu(A) < \epsilon$.

   Proof: Suppose $\mu \ll \nu$, the proof is done by contradiction. Suppose there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists a measurable set A such that $\nu(A) < \delta$ but $\mu(A) \geq \epsilon$. By our assumption, for each $n \geq 1$ there exists a measurable subset A_n such that $\nu(A_n) < \dfrac{1}{2^n}$ and $\mu(A_n) \geq \epsilon$. Let $A = \limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$.

Since $\sum_{n=1}^{\infty} \nu(A_n) < \infty$, then by Borel-Cantelli Lemma (problem 3(c) in Exercises 7) we have $\nu(A) = 0$. But then $\mu(A) = 0$. Since μ is a finite measure, by problem 3(b) in Exercises 7, we have

$$0 = \mu(A) = \mu(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mu(A_n) \geq \epsilon,$$

a contradiction. Therefore, for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{B}$ with $\nu(A) < \delta$ then $\mu(A) < \epsilon$.