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Measure and Integration Solutions 2

1. Let $a < s < b$, and suppose $f : [a, b] \to \mathbb{R}$ is bounded and continuous at s. Let $\Psi : [a, b] \to R$ be given by

$$\Psi(x) = \begin{cases} 0 & \text{if } a \leq x \leq s \\ 1 & \text{if } s < x \leq b. \end{cases}$$

Show that f is Ψ-Riemann integrable, and $\int_a^b f(x)d\Psi(x) = f(s)$.

Proof. Let $\epsilon > 0$. By continuity of f at s there exists a $\delta > 0$ such that $|f(x) - f(s)| < \epsilon$ for $|x - s| < \delta$, $x \in [a, b]$. Let $\mathcal{C}$ be a finite exact non-overlapping cover of $[a, b]$ with $\|\mathcal{C}\| < \delta$. Notice that $\Delta_I \Psi = 0$ for all $I \in \mathcal{C}$ with $s \notin I$. The point s belongs to at most two elements of $\mathcal{C}$. If s is a right end-point of some I, then $\Delta_I \Psi = 0$. If s is a left end-point or an interior point of some I, then $\Delta_I \Psi = 1$. Let I_0 be the unique element of $\mathcal{C}$ containing s as a left end-point or an interior point, then for any choice function ξ,

$$\mathcal{R}(f|\Psi; \mathcal{C}, \xi) = f(\xi(I_0)).$$

Since $\|\mathcal{C}\| < \delta$, then

$$|\mathcal{R}(f|\Psi; \mathcal{C}, \xi) - f(s)| = |f(\xi(I_0)) - f(s)| < \epsilon.$$

Thus, f is Ψ-Riemann integrable, and $\int_a^b f(x)d\Psi(x) = f(s)$.

2. Let $a = a_0 < a_1 < a_2 < \cdots < a_n = b$, and suppose that the function $\Psi : [a, b] \to R$ has the constant value c_i on the interval (a_{i-1}, a_i) for $i = 1, 2, \cdots, n$. Show that if $f : [a, b] \to \mathbb{R}$ is continuous, then f is Ψ-Riemann integrable, and

$$\int_a^b f(x)d\Psi(x) = \sum_{i=0}^{n} f(a_i)d_i,$$

where

$$d_i = \begin{cases} c_1 - \Psi(a) & \text{if } i = 0 \\ c_{i+1} - c_i & \text{if } 1 \leq i \leq n-1 \\ \Psi(b) - c_n & \text{if } i = n. \end{cases}$$

Proof. Let $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in [a, b]$ with $|x - y| < \delta$. Let $\mathcal{C}$ be any finite exact non-overlapping cover of $[a, b]$ with $\|\mathcal{C}\| < \delta$. Assume with no loss of generality that the $a_i's$ are end-points of some of the intervals in $\mathcal{C}$ (otherwise we refine $\mathcal{C}$ further). Call these intervals $I_0, I_1^-, I_1^+, \ldots, I_{n-1}^-, I_{n-1}^+, I_n$, where I_0 contains $a = a_0$ as a left end point, I_n contains $a_n = b$ as a right end-point, and for $i = 1, 2, \ldots, n-1$, I_i^- contains a_i as a right end-point, and I_i^+ contains a_i as a left end-point. Notice that if $I \in \mathcal{C}$ with $a_i \notin I$ for all $i = 0, 1, \ldots, n$, then $\Delta_I \Psi = 0$. Thus, for any choice function ξ we have,

$$\mathcal{R}(f|\Psi; \mathcal{C}, \xi) = f(\xi(I_0))\Delta_{I_0}\Psi + \sum_{i=1}^{n-1}(f(\xi(I_i^-)\Delta_{I_i^-}\Psi + f(\xi(I_i^+))\Delta_{I_i^+}\Psi) + f(\xi(I_n))\Delta_{I_n}\Psi.$$

Now, $\Delta_{I_0}\Psi = c_1 - \Psi(a)$, $\Delta_{I_n}\Psi = \Psi(b) - c_n$ and for $i = 1, 2, \ldots, n-1$, $\Delta_{I_i^-}\Psi = \Psi(a_i) - c_i$ and $\Delta_{I_i^+}\Psi = c_{i+1} - \Psi(a_i)$. Notice that $d_i = (c_{i+1} - \Psi(a_i)) - (\Psi(a_i) - c_i)$ for $i = 1, 2, \ldots, n-1$. Thus,

$$\begin{aligned} |\mathcal{R}(f|\Psi; \mathcal{C}, \xi) - \sum_{i=0}^{n} f(a_i)d_i| &\leq |f(\xi(I_0) - f(a)||c_1 - \Psi(a)| \\ &+ \sum_{i=1}^{n-1} |f(\xi(I_i^-) - f(a_i)||\Psi(a_i) - c_i| \\ &+ \sum_{i=1}^{n-1} |f(\xi(I_i^+) - f(a_i)||\Psi(a_i) - c_{i+1}| \\ &+ |f(\xi(I_n) - f(b)||\Psi(b) - c_n| \\ &< \epsilon M, \end{aligned}$$

where $M = |c_1 - \Psi(a)| + \sum_{i=1}^{n-1} |(\Psi(a_i) - c_i| + \sum_{i=1}^{n-1} |(\Psi(a_i) - c_{i+1}| + |\Psi(b) - c_n|$. Thus, f is Ψ-Riemann integrable, and $\int_a^b f(x)d\Psi(x) = \sum_{i=0}^{n} f(a_i)d_i$.

3. Let $\Psi : [a, b] \to R$ be non-decreasing, and let $f : [a, b] \to \mathbb{R}$ be bounded. Show that f is Ψ-Riemann integrable **if and only if** for every $\epsilon > 0$, there exists a $\delta > 0$ such that
$$\sum_{\{I \in \mathcal{C} : \sup_{x \in I} f(x) - \inf_{x \in I} f(x) \geq \epsilon\}} \Delta_I \Psi < \epsilon$$
for all finite non-overlapping exact covers $\mathcal{C}$ of $[a, b]$ such that $||\mathcal{C}|| < \delta$.

Proof. Suppose f is Ψ-Riemann integrable, then
$$\lim_{\|\mathcal{C}\| \to 0} \mathcal{U}(f|\Psi; \mathcal{C}) = \lim_{\|\mathcal{C}\| \to 0} \mathcal{L}(f|\Psi; \mathcal{C}) = \int_{[a,b]} f(x)d\Psi(x).$$
Thus, given $\epsilon > 0$ there exists a $\delta > 0$ such that
$$\mathcal{U}(f|\Psi; \mathcal{C}) - \mathcal{L}(f|\Psi; \mathcal{C}) < \epsilon^2$$
for all finite exact non-overlapping covers of $[a, b]$ such that $\|\mathcal{C}\| < \delta$. Let $\mathcal{B} = \{I \in \mathcal{C} : \sup_{x \in I} f(x) - \inf_{x \in I} f(x) \geq \epsilon\}$. Then,
$$\begin{aligned} \epsilon \sum_{I \in \mathcal{B}} \Delta_I \Psi &\leq \sum_{I \in \mathcal{B}} \sup_{x \in I} f(x) - \inf_{x \in I} f(x)\Delta_I \Psi \\ &\leq \sum_{I \in \mathcal{C}} \sup_{x \in I} f(x) - \inf_{x \in I} f(x)\Delta_I \Psi \\ &= \mathcal{U}(f|\Psi; \mathcal{C}) - \mathcal{L}(f|\Psi; \mathcal{C}) < \epsilon^2. \end{aligned}$$

This implies that $\sum_{I\in\mathcal{B}} \Delta_I\Psi < \epsilon$.

Conversely, let $\epsilon > 0$. By hypothesis, there exists a $\delta > 0$ such that

$$\sum_{\{I\in\mathcal{C}\,:\,\sup_{x\in I} f(x)-\inf_{x\in I} f(x)\geq\epsilon\}} \Delta_I\Psi < \epsilon$$

for all finite non-overlapping exact covers $\mathcal{C}$ of $[a,b]$ such that $||\mathcal{C}|| < \delta$. Let $\mathcal{B} = \{I \in \mathcal{C} : \sup_{x\in I} f(x) - \inf_{x\in I} f(x) \geq \epsilon\}$ and $\mathcal{G} = \mathcal{C} \setminus \mathcal{B}$. Set $M = \sup_{x\in[a,b]} |f(x)|$, then

$$\begin{aligned}
\mathcal{U}(f|\Psi;\mathcal{C}) - \mathcal{L}(f|\Psi;\mathcal{C}) &= \sum_{I\in\mathcal{B}} \sup_{x\in I} f(x) - \inf_{x\in I} f(x)\Delta_I\Psi \\
&+ \sum_{I\in\mathcal{G}} \sup_{x\in I} f(x) - \inf_{x\in I} f(x)\Delta_I\Psi \\
&< 2M\epsilon + \epsilon(\Psi(b) - \Psi(a)) \\
&= \epsilon(2M + \Psi(b) - \Psi(a)).
\end{aligned}$$

Thus, f is Ψ-Riemann integrable.

4. Let $\Psi : [a,b] \to R$ be non-decreasing, and $f : [a,b] \to \mathbb{R}$ be bounded. Show that if f is Ψ-Riemann integrable, then the function $f^2; [a,b] \to \mathbb{R}$ given by $f^2(x) = (f(x))^2$ is Ψ-Riemann integrable.

Proof. Let $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$. Consider the function $g : [m,M] \to \mathbb{R}$ defined by $g(x) = x^2$. Notice that g is uniformly continuous on $[m,M]$, and $f^2(x) = g(f(x)) = g\circ f(x)$. Let $\epsilon > 0$, there exists $0 < \delta < \epsilon$ such that $|g(u)-g(v)| < \epsilon$ for all $u, v \in [m,M]$ with $|u-v| < \delta$. Since f is Ψ-Riemann integrable, there exists a $\delta' > 0$ such that $\mathcal{U}(f|\Psi;\mathcal{C}) - \mathcal{L}(f|\Psi;\mathcal{C}) < \delta^2$ for all finite exact non-overlapping covers of $[a,b]$ with $\|\mathcal{C}\| < \delta'$. Let $\mathcal{C}$ be a finite exact non-overlapping covers of $[a,b]$ with $\|\mathcal{C}\| < \delta'$. Define

$$\mathcal{G} = \{I \in \mathcal{C} : \sup_{x\in I} f(x) - \inf_{x\in I} f(x) < \delta\} \text{ and } \mathcal{B} = \{I \in \mathcal{C} : \sup_{x\in I} f(x) - \inf_{x\in I} f(x) \geq \delta\}.$$

Notice that if $I \in \mathcal{G}$, then $\sup\limits_{x\in I} g(f(x)) - \inf\limits_{x\in I} g(f(x)) \leq \epsilon$. Now,

$$\begin{aligned}
\delta\sum_{I\in\mathcal{B}} \Delta_I\Psi &\leq \sum_{I\in\mathcal{B}} (\sup_{x\in I} f(x) - \inf_{x\in I} f(x))\Delta_I\Psi \\
&\leq \sum_{I\in\mathcal{C}} (\sup_{x\in I} f(x) - \inf_{x\in I} f(x))\Delta_I\Psi \\
&= \mathcal{U}(f|\Psi;\mathcal{C}) - \mathcal{L}(f|\Psi;\mathcal{C}) < \delta^2.
\end{aligned}$$

Thus, $\sum\limits_{I\in\mathcal{B}} \Delta_I\Psi < \delta < \epsilon$. Finally,

$$\begin{aligned}
\mathcal{U}(f^2|\Psi;\mathcal{C}) - \mathcal{L}(f^2|\Psi;\mathcal{C}) &= \mathcal{U}(g\circ f|\Psi;\mathcal{C}) - \mathcal{L}(g\circ f|\Psi;\mathcal{C}) \\
&= \sum_{I\in\mathcal{G}} (\sup_{x\in I} g(f(x)) - \inf_{x\in I} g(f(x)))\Delta_I\Psi \\
&+ \sum_{I\in\mathcal{B}} (\sup_{x\in I} g(f(x)) - \inf_{x\in I} g(f(x)))\Delta_I\Psi \\
&< \epsilon(\Psi(b) - \Psi(a)) + 2\|g\|_u),
\end{aligned}$$

where $\|g\|_u = \sup_{u\in[m,M]} |g(u)|$. Thus, $\lim_{\|\mathcal{C}\|\to 0} \mathcal{U}(f^2|\Psi;\mathcal{C}) - \mathcal{L}(f^2|\Psi;\mathcal{C}) = 0$ and f^2 is Ψ-Riemann integrable.