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Measure and Integration Solutions 3

1. Suppose $\Psi : [a,b] \to \mathbb{R}$ is non-decreasing, and $f : [a,b] \to \mathbb{R}$ is Ψ-Riemann integrable. Assume that $m < M$, and $m \leq f(x) \leq M$. Let $g : [m,M] \to \mathbb{R}$ be continuous. Show that the function $g \circ f : [a,b] \to \mathbb{R}$ is Ψ-Riemann integrable.

   Proof Let $\epsilon > 0$, by uniform continuity of g there exists $0 < \delta < \epsilon$ so that $|g(u) - g(v)| < \epsilon$ for all $u, v \in [m,M]$ with $|u - v| < \delta$. Since f is Ψ-Riemann integrable, there exists $\delta' > 0$ so that $\mathcal{U}(f;\mathcal{C}) - \mathcal{L}(f;\mathcal{C}) < \delta^2$ for all finite non-overlapping exact covers $\mathcal{C}$ of $[a,b]$ such that $||\mathcal{C}|| < \delta'$. For any $\mathcal{C}$ with $||\mathcal{C}|| < \delta'$, let $\mathcal{A} = \{I \in \mathcal{C} : \sup_{x\in I} f(x) - \inf_{x\in I} f(x) < \delta\}$, and $\mathcal{B} = \mathcal{C} \setminus \mathcal{A}$. Notice that if $I \in \mathcal{A}$, then $\sup_{x\in I} g(f(x)) - \inf_{x\in I} g(f(x)) \leq \epsilon$. Furthermore,

$$\begin{aligned} \delta \sum_{I\in\mathcal{B}} \Delta_I \Psi &\leq \sum_{I\in\mathcal{B}} (\sup_{x\in I} f(x) - \inf_{x\in I} f(x))\Delta_I \Psi \\ &\leq \sum_{I\in\mathcal{C}} (\sup_{x\in I} f(x) - \inf_{x\in I} f(x))\Delta_I \Psi \\ &= \mathcal{U}(f;\mathcal{C}) - \mathcal{L}(f;\mathcal{C}) < \delta^2. \end{aligned}$$

   Thus, $\sum_{I\in\mathcal{B}} \Delta_I \Psi < \delta$, and,

$$\begin{aligned} \mathcal{U}(g\circ f;\mathcal{C}) - \mathcal{L}(g\circ f;\mathcal{C}) &= \sum_{I\in\mathcal{A}} \left(\sup_{x\in I} g(f(x)) - \inf_{x\in I} g(f(x))\right)\Delta_I\Psi \\ &+ \sum_{I\in\mathcal{B}} \left(\sup_{x\in I} g(f(x)) - \inf_{x\in I} g(f(x))\right)\Delta_I\Psi \\ &< \epsilon(\Psi(b) - \Psi(a)) + 2||g||_u\delta < \epsilon(\Psi(b) - \Psi(a) + 2||g||_u). \end{aligned}$$

   Therefore, $g \circ f$ is Ψ-Riemann integrable.

2. Let $A, B \subseteq \mathbb{R}^N$. Prove the following.

   (a) If $|A|_e = 0$, then $|A \cup B|_e = |B|_e$.

   (b) If $|A\Delta B|_e = 0$, then $|A \cup B|_e = |A|_e = |B|_e = |A \cap B|_e$.

   Proof (a) Suppose $|A|_e = 0$, then $|B|_e \leq |A \cup B|_e \leq |A|_e + |B|_e = |B|_e$. Thus, $|A \cup B|_e = |B|_e$.

   Proof (b) Suppose $|A\Delta B|_e = 0$. Since $A\cup B = (A\Delta B)\cup(A\cap B)$, it follows from part (a) that $|A\cup B|_e = |A\cap B|_e$. Furthermore, since $A\Delta B = (A\setminus B)\cup(B\setminus A)$, it follows that $|A\setminus B|_e = |B\setminus A|_e = 0$. Now $A = (A\setminus B)\cup(A\cap B)$ and $B = (B\setminus A)\cup(A\cap B)$, hence by part (a) $|A|_e = |B|_e = |A\cap B|_e = |A\cup B|_e$.

3. Let K_1, K_2 be compact subsets of $\mathbb{R}^n$ such that $K_1 \cap K_2 = \emptyset$. Show that

$$|K_1 \cup K_2|_e = |K_1|_e + |K_2|_e.$$

Proof By Lemma 2.1.2, it is enough to show that $\text{dist}(K_1, K_2) > 0$. The proof is done by contradiction. Suppose that $\text{dist}(K_1, K_2) = 0$, then there exist sequences $(x_n) \subseteq K_1$ and $(y_n) \subseteq K_2$ such that $|x_n - y_n| \to 0$. Since K_1 is compact, then the sequence (x_n) has a convergent subsequence (x_{n_j}) converging to say $x \in K_1$. But then,

$$|y_{n_j} - x| \le |y_{n_j} - x_{n_j}| + |x_{n_j} - x| \to 0.$$

Thus, $(y_{n_j}) \to x$. Since $(y_{n_j}) \subseteq K_2$ and K_2 is closed, then $x \in K_2$. Hence, $x \in K_1 \cap K_2$, which is a contradiction to the assumption that $K_1 \cap K_2 = \emptyset$. Thus, $\text{dist}(K_1, K_2) = 0$, and by lemma 2.1.2

$$|K_1 \cup K_2|_e = |K_1|_e + |K_2|_e.$$

4. Let F be a closed subset of $\mathbb{R}^N$. For each $n \ge 1$, let

$$G_n = \{x \in \mathbb{R}^N : |x - y| < \frac{1}{n} \text{ for some } y \in F\}.$$

(a) Show that G_n is open for each $n \ge 1$.

(b) Show that $F = \bigcap_{n=1}^{\infty} G_n$.

(c) Conclude that $\mathcal{F} \subseteq \mathcal{O}_\delta$. Here $\mathcal{F}$ denotes the collection of all closed subset of $\mathbb{R}^N$, and $\mathcal{O}_\delta$ denotes the collections of all subsets of $\mathbb{R}^N$ that can be written as the countable intersection of open setsets of $\mathbb{R}^N$.

Proof (a) Let $H_n = \mathbb{R}^N \setminus G_n$, then $H_n = \{x \in \mathbb{R}^N : \inf_{y \in F} |x - y| \ge \frac{1}{n}\}$. We show that H_n is closed. To this end, let (x_m) be a sequence in H_n converging to $x \in \mathbb{R}^N$. We must show that $x \in H_n$. Let $\epsilon > 0$, then there exists an integer $M > 0$ such that $|x_m - x| < \epsilon$ for all $m \ge M$. Pick any $m \ge M$, then for all $y \in F$

$$|x - y| \ge |x_m - y| - |x - x_m| \ge \frac{1}{n} - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $|x-y| \ge \frac{1}{n}$ for all $y \in F$, i.e. $\inf_{y \in F} |x-y| \ge \frac{1}{n}$. Thus, $x \in H_n$ which implies that H_n is closed, and hence, G_n is open.

Proof (b) Clearly, $F \subseteq G_n$ for all n, hence $F \subseteq \bigcap_{n=1}^{\infty} G_n$. Now suppose that $x \in \bigcap_{n=1}^{\infty} G_n$, then for each n there exists $y_n \in F$ such that $|x - y_n| < 1/n$. Then, (y_n) is a sequence in F converging to x Since F is closed, this implies that $x \in F$. Thus, $\bigcap_{n=1}^{\infty} G_n \subseteq F$. Therefore, $F = \bigcap_{n=1}^{\infty} G_n$.

Proof (c) By parts (a) and (b), each closed set F is of the form $\bigcap_{n=1}^{\infty} G_n$ with G_n open i.e. $\bigcap_{n=1}^{\infty} G_n \in \mathcal{O}_\delta$.