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Measure and Integration Solutions 4

1. Let $A, B \subseteq \mathbb{R}^N$, and suppose that $A \subseteq B$ and $|B \setminus A|_e = 0$. Show that if A is measurable, then B is measurable and $|A| = |B|$.

   Proof Since $|B \setminus A|_e = 0$, then $B \setminus A$ is measurable, and $B = A \cup (B \setminus A)$ is a union of two measurable sets, hence B is measurable. Furthermore, $|A| \leq |B| \leq |A| + |B \setminus A| = |A|$. Thus, $|A| = |B|$.

2. Prove that $|x + E|_e = |E|_e$ for all $x \in \mathbb{R}^N$ and every $E \subseteq \mathbb{R}^N$.

   Proof First notice that if $I = \Pi_{k=1}^N [a_k, b_k]$ is a rectangle, and $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, then $I + x = \Pi_{k=1}^N [a_k + x_k, b_k + x_k]$ and $I - x = \Pi_{k=1}^N [a_k - x_k, b_k - x_k]$ are also rectangles, and $\mathrm{v}ol(I) = \mathrm{v}ol(I + x) = \mathrm{v}ol(I - x)$. Now, let $\mathcal{C} = \{I_n\}$ be any countable cover of E by rectangles, then $\mathcal{D} = \{I_n + x\}$ is a countable cover of $E + x$. Thus,

   $$|E + x|_e \leq \sum_{n=1}^{\infty} \mathrm{v}ol(I_n + x) = \sum_{n=1}^{\infty} \mathrm{v}ol(I_n).$$

   Since $\mathcal{C}$ was an arbitrary cover of E, it follows that $|x + E|_e \leq |E|_e$. The other inequality is proved similarly by starting with a countable cover of $E + x$ by rectangles.

3. Let $A \subseteq \mathbb{R}^M$. The *inner Lebesgue measure* of A is defined by

   $$|A|_i = \sup\{|K|_e : K \subseteq A,\ K \text{ is compact }\}.$$

   Prove the following.

   (a) $|A|_i \leq |A|_e$ for all $A \in \mathbb{R}^M$.

   (b) If $A \subseteq B$, then $|A|_i \leq |B|_i$.

   (c) If $A_1, A_2, \ldots$ are disjoint, then $|\bigcup_{n=1}^{\infty} A_n|_i \geq \sum_{n=1}^{\infty} |A_n|_i$.

   (d) If A is compact or open, then $|A|_e = |A|_i$.

   Proof (a) Let G be any open subset containing A. Then, $|K|_e \leq |G|_e = |G|$ for any compact subset K of A. Hence $|A|_i \leq |G|_e$. Since G is an arbitrary open set containing A, it follows that $|A|_i \leq |A|_e$.

   Proof (b) Follows from the fact that any compact subset K of A is also a compact subset of B.

Proof (c) Fix an integer N, and let $K_1, K_2, \ldots, K_N$ be any compact subsets of $A_1, A_2, \ldots, A_N$ respectively. Notice that $K_1, K_2, \ldots, K_N$ are pairwise disjoint, and $\bigcup_{j=1}^{N} K_j$ is a compact subset of $\bigcup_{n=1}^{\infty} A_n$. Then,

$$\sum_{j=1}^{N} |K_j|_e = |\bigcup_{j=1}^{N} K_j|_e \leq |\bigcup_{n=1}^{\infty} A_n|_i.$$

Taking first the supremum over compact subsets $K_1 \subseteq A_1$, then the supremum over compact subsets $K_2 \subseteq A_2$, and so on and finally the supremum over compact subsets $K_N \subseteq A_N$, one gets

$$\sum_{j=1}^{N} |A_j|_i \leq |\bigcup_{n=1}^{\infty} A_n|_i.$$

Finally taking the limit as $N \to \infty$, one has

$$\sum_{j=1}^{\infty} |A_j|_i \leq |\bigcup_{n=1}^{\infty} A_n|_i.$$

Proof (d) Suppose that A is compact. Since $A \subseteq A$, then

$$|A|_e \leq \sup\{|K|_e : K \subseteq A,\ K \text{ is compact }\} = |A|_i.$$

Combining this with part (a), one gets that $|A|_e = |A|_i$.

Suppose that A is open. By Lemma 2.1.9, there exists a countable non-overlapping exact cover of A by cubes Q_n, i.e., $A = \bigcup_{n=1}^{\infty} Q_n$, and hence $|A|_e \leq \sum_{n=1}^{\infty} |Q_n|_e$. For any fixed N, $\bigcup_{n=1}^{N} Q_n$ is a compact subset of A, hence by Lemma 2.1.1

$$\sum_{n=1}^{N} |Q_n|_e = |\bigcup_{n=1}^{N} Q_n|_e \leq |A|_i.$$

Taking the limit as $N \to \infty$, we have

$$|A|_e \leq \sum_{n=1}^{\infty} |Q_n|_e \leq |A|_i.$$

By part (a), we have $|A|_e = |A|_i$.