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Measure and Integration Solutions 5

1. Suppose $A_1, A_2 \subseteq \mathbb{R}^N$ are Lebesgue measurable.
   (a) Show that if $A_1 \subseteq A_2$ and $|A_1| < \infty$, then $|A_2 \setminus A_1| = |A_2| - |A_1|$.
   (b) Show that if $|A_1 \cap A_2| < \infty$, then $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

   Proof (a) $A_2 = A_1 \cup (A_2 \setminus A_1)$ (a disjoint union), hence

   $$|A_2| = |A_1| + |A_2 \setminus A_1|.$$

   If $|A_1| < \infty$, then we can subtract $|A_1|$ from both sides leading to

   $$|A_2 \setminus A_1| = |A_2| - |A_1|.$$

   Proof (b) First notice that $A_1 \cap A_2 \subseteq A_2$ and $|A_1 \cap A_2| < \infty$, hence by part (a)

   $$|A_2 \setminus (A_1 \cap A_2)| = |A_2| - |A_1 \cap A_2|.$$

   Now, $A_1 \cup A_2 = A_1 \cup (A_2 \setminus (A_1 \cap A_2))$ (a disjoint union), hence

   $$|A_1 \cup A_2| = |A_1| + |A_2 \setminus (A_1 \cap A_2)| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

2. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}^N$.
   (a) Show that if $|\Gamma_n \cap \Gamma_m| = 0$ for $n \neq m$, then $|\bigcup_{n=1}^{\infty} \Gamma_n| = \sum_{n=1}^{\infty} |\Gamma_n|$.
   (b) Show that if $\Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$, then $|\bigcup_{n=1}^{\infty} \Gamma_n| = \lim_{n\to\infty} |\Gamma_n|$.
   (c) Show that if if $|\Gamma_1| < \infty$ and $\Gamma_1 \supseteq \Gamma_2 \supseteq \ldots$, then $|\bigcap_{n=1}^{\infty} \Gamma_n| = \lim_{n\to\infty} |\Gamma_n|$.

   Proof (a) Let $\Gamma_0 = \emptyset$, $A_1 = \Gamma_1$, $B_1 = \emptyset$. For $n \geq 2$, set $A_n = \Gamma_n \setminus \bigcup_{m=1}^{n-1} \Gamma_m$ and $B_n = \Gamma_n \cap \bigcup_{m=1}^{n-1} \Gamma_m = \bigcup_{m=1}^{n-1} (\Gamma_n \cap \Gamma_m)$. Then,
   – $\Gamma_n = A_n \cup B_n$ for all $n \geq 1$,
   – $A_n \cap A_m = \emptyset$ for $m \neq n$,
   – $|B_n| = 0$ for all $n \geq 1$ (since $|\Gamma_n \cap \Gamma_m| = 0$ for $n \neq m$), hence $|\Gamma_n| = |A_n|$ for all $n \geq 1$,
   – $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \Gamma_n$: clearly the left handside is a subset of the right handside. Now, let $x \in \bigcup_{n=1}^{\infty} \Gamma_n$, then $x \in \Gamma_n$ for some n. Let n_0 be the smallest positive integer such that $x \in \Gamma_{n_0}$, then $x \in A_{n_0} \subseteq \bigcup_{n=1}^{\infty} A_n$.

Hence,

$$|\bigcup_{n=1}^{\infty}\Gamma_n| = |\bigcup_{n=1}^{\infty}A_n| = \sum_{n=1}^{\infty}|A_n| = \sum_{n=1}^{\infty}|\Gamma_n|.$$

Proof (b) Let $\Gamma_0 = \emptyset$ and $A_n = \Gamma_n \setminus \Gamma_{n-1}$ for $n \geq 1$. Then,

– $A_n \cap A_m = \emptyset$ for $m \neq n$,

– $\Gamma_n = \bigcup_{m=1}^{n} A_n$ for all $n \geq 1$,

– $\bigcup_{n=1}^{\infty}\Gamma_n = \bigcup_{n=1}^{\infty}A_n$.

Hence,

$$\lim_{n\to\infty}|\Gamma_n| = \lim_{n\to\infty}|\bigcup_{m=1}^{n}A_m| = \lim_{n\to\infty}\sum_{m=1}^{n}|A_m| = \sum_{m=1}^{\infty}|A_m| = |\bigcup_{n=1}^{\infty}A_n| = |\bigcup_{n=1}^{\infty}\Gamma_n|.$$

Proof (c) Let $E_n = \Gamma_1 \setminus \Gamma_n$ for $n \geq 1$. Then,

– $E_1 \subseteq E_2 \subseteq \dots$,

– $|E_n| = |\Gamma_1| - |\Gamma_n|$ (since $\Gamma_n \subseteq \Gamma_1$ and $|\Gamma_n| < \infty$, see part (a) of exercise 1),

– $\bigcup_{n=1}^{\infty}E_n = \Gamma_1 \setminus \bigcap_{n=1}^{\infty}\Gamma_n$, and hence $|\bigcup_{n=1}^{\infty}E_n| = |\Gamma_1| - |\bigcap_{n=1}^{\infty}\Gamma_n|$ (since $|\bigcap_{n=1}^{\infty}\Gamma_n| < \infty$),

By part (b),

$$|\bigcup_{n=1}^{\infty}E_n| = \lim_{n\to\infty}E_n.$$

Hence, $|\bigcap_{n=1}^{\infty}\Gamma_n| = \lim_{n\to\infty}|\Gamma_n|$.

3. Let $A \subseteq \mathbb{R}^N$ be Lebesgue measurable. Show that there exists a sequence $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$ of compact subsets of A such that $|A \setminus \bigcup_{n=1}^{\infty}K_n| = 0$.

Proof Since A is measurable, then A^c is also measurable. For each $n \geq 1$, there exists an open subset G_n such that $A^c \subseteq G_n$ and $|G_n \setminus A^c| < 1/n$. Let $F_n = G_n^c$, then F_n is a closed subset of A and $|A \setminus F_n| = |G_n \setminus A^c| < 1/n$. For $n \geq 1$, let $K_n = \bigcup_{m=0}^{n} F_m \cap \overline{B(0,n)}$, where $\overline{B(0,n)}$ is the closed ball with centre the origin and radius n. Then,

– K_n is a compact subset of A (since it is closed and bounded),

– $K_1 \subseteq K_2 \subseteq \dots$,

– $\bigcup_{n=0}^{\infty}F_n = \bigcup_{n=0}^{\infty}K_n$: clearly the right handside is contained in the left handside. Now, let $x \in \bigcup_{n=0}^{\infty}F_n$, then $x \in F_n$ for some n. Also, there exists an integer m such that $x \in \overline{B(0,m)}$. If $m \leq n$, then $x \in K_n \subseteq \bigcup_{n=0}^{\infty}K_n$, and if $m > n$, then $x \in K_m \subseteq \bigcup_{n=0}^{\infty}K_n$.

Thus, for each $n \geq 1$,

$$|A \setminus \bigcup_{n=1}^{\infty}K_n| = |A \setminus \bigcup_{n=1}^{\infty}F_n| \leq |A \setminus F_n| < 1/n.$$

Taking the limit as $n \to \infty$, one gets $|A \setminus \bigcup_{n=1}^{\infty}K_n| = 0$.