Universiteit Utrecht — Universiteit Utrecht — Boedapestlaan 6

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Measure and Integration Exercises 6

1. Let E be an uncountable set, and $\mathcal{B} = \{A \subseteq E : A \text{ or } A^c \text{ is countable}\}$. Show that $\mathcal{B}$ is a σ-algebra over E.

   Proof It is clear that $\emptyset \in \mathcal{B}$, and if $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$. We need to show that $\mathcal{B}$ is closed under countable unions. Let $A_1, A_2, \ldots \in \mathcal{B}$, If A_n is countable for each n, then $\bigcup_{n=1}^\infty A_n$ is countable, and hence $\bigcup_{n=1}^\infty A_n \in \mathcal{B}$. Suppose there exists an i such that A_i is uncountable. Since $A_i \in \mathcal{B}$, then A_i^c must be countable. Now, $(\bigcup_{n=1}^\infty A_n)^c = \bigcap_{n=1}^\infty A_n^c \subseteq A_i^c$. Thus, $(\bigcup_{n=1}^\infty A_n)^c$ is countable and hence $\bigcup_{n=1}^\infty A_n \in \mathcal{B}$. This shows that $\mathcal{B}$ is a σ-algebra over E.

2. Let E be a set, and $\mathcal{C} \subseteq \mathcal{P}(E)$. Consider $\sigma(E;\mathcal{C})$, the smallest σ-algebra over E containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(E;\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(E;\mathcal{C}_0)$.

   (a) Show that $\mathcal{D}$ is a σ-algebra over E.

   (b) Show that $\mathcal{D} = \sigma(E;\mathcal{C})$.

   Proof (a): Clearly $\emptyset \in \mathcal{D}$ since $\emptyset$ belongs to every σ-algebra. Let $A \in \mathcal{D}$, then there is a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $A \in \sigma(E;\mathcal{C}_0)$. But then $A^c \in \sigma(E;\mathcal{C}_0)$, hence $A^c \in \mathcal{D}$. Finally, let $\{A_n\}$ be in $\mathcal{D}$, then for each n there exists a countable collection $\mathcal{C}_n \subseteq \mathcal{C}$ such that $A_n \in \sigma(E;\mathcal{C}_n)$. Let $\mathcal{C}_0 = \bigcup_n \mathcal{C}_n$, then $\mathcal{C}_0 \subseteq \mathcal{C}$, and $\mathcal{C}_0$ is countable. Furthermore, $\sigma(E;\mathcal{C}_n) \subseteq \sigma(E;\mathcal{C}_0)$, and hence $A_n \in \sigma(E;\mathcal{C}_0)$ for each n which implies that $\bigcup_n A_n \in \sigma(E;\mathcal{C}_0)$. Therefore, $\bigcup_n A_n \in \mathcal{D}$ and $\mathcal{D}$ is a σ-algebra.

   Proof (b): By definition $\mathcal{D} \subseteq \sigma(E;\mathcal{C})$. Also, $\mathcal{C} \subseteq \mathcal{D}$ since $C \in \sigma(E;\{C\})$ for every $C \in \mathcal{C}$. Since $\sigma(E;\mathcal{C})$ is the smallest σ-algebra over E containg $\mathcal{C}$, then by part (a) $\sigma(E;\mathcal{C}) \subseteq \mathcal{D}$. Thus, $\mathcal{D} = \sigma(E;\mathcal{C})$.

3. A collection $\mathcal{M}$ of sets is said to be a *monotone class* if it satisfies the following two properties:

   (i) if $\{A_n\} \subseteq \mathcal{M}$ with $A_1 \subseteq A_2 \subseteq \ldots$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{M}$, and

   (ii) if $\{B_n\} \subseteq \mathcal{M}$ with $B_1 \supseteq B_2 \supseteq \ldots$, then $\bigcap_{n=1}^\infty B_n \in \mathcal{M}$.

   (a) Show that the intersection of an arbitrary collection of monotone classes is a monotone class.

   (b) Let E be a set, and $\mathcal{B}$ a collection of subsets of E. Show that $\mathcal{B}$ is a σ-algebra if and only if $\mathcal{B}$ is an algebra and a monotone class.

(c) Let $\mathcal{A}$ be an algebra over E, and $\mathcal{M}$ the smallest monotone class containing $\mathcal{A}$, i.e. $\mathcal{M}$ is the intersection of all monotone classes containing $\mathcal{A}$. Show that $\mathcal{M}$ is an algebra.

(d) Using the same notation as in part (c), show that $\mathcal{M} = \sigma(E, \mathcal{A})$, where $\sigma(E, \mathcal{A})$ is the smallest σ-algebra over E containg the algebra $\mathcal{A}$.

Proof(a) EASY!

Proof(b) It is clear that any σ-algebra is both an algebra and a monotone class. Suppose that $\mathcal{B}$ is an algebra and a monotone class. We only need to show that $\mathcal{B}$ is closed under countable unions. Let $A_1, A_2, \ldots,$ be a sequence in $\mathcal{B}$. Set $B_n = \cup_{m=1}^n A_n$, then $\cup_{n=1}^\infty A_n = \cup_{n=1}^\infty B_n$, and $B_1 \subseteq B_2 \subseteq \ldots$. Since $\mathcal{B}$ is an algebra, then $B_n \in \mathcal{B}$, and since $\mathcal{B}$ is a monotone class, then $\cup_{n=1}^\infty A_n = \cup_{n=1}^\infty B_n \in \mathcal{B}$.

Proof(c) We first show that $\mathcal{M}_1 = \{B \subseteq E : B^c, B \cup A \in \mathcal{M} \text{ for all } A \in \mathcal{A}\}$ is a monotone class containing $\mathcal{A}$. Clearly, $\mathcal{A} \subseteq \mathcal{M}_1$. Now let $\{B_n\} \in \mathcal{M}_1$ with $B_1 \subseteq B_2 \subseteq, \ldots$. Then,

— $B_n^c, B_n \cup A \in \mathcal{M}$ for all $n \geq 1$ and $A \in \mathcal{A}$,

— $B_1^c \supseteq B_2^c \supseteq \ldots$

— $B_1 \cup A \subseteq B_2 \cup A \subseteq, \ldots$ for all $A \in \mathcal{A}$.

Since $\mathcal{M}$ is a monotone class, then $(\bigcup_{n=1}^\infty B_n)^c = \bigcap_{n=1}^\infty B_n^c \in \mathcal{M}$. Further, for all $A \in \mathcal{A}$, one has $(\bigcup_{n=1}^\infty B_n) \cup A = \bigcup_{n=1}^\infty (B_n \cup A) \in \mathcal{M}$. Thus, $\bigcup_{n=1}^\infty B_n \in \mathcal{M}$, and $\mathcal{M}_1$. Similarly one can show that if $C_1 \supseteq C_2 \supseteq \ldots$ is a sequence in $\mathcal{M}$, then $\bigcap_{n=1}^\infty C_n \in \mathcal{M}$. Thus, $\mathcal{M}_1$ is a monotone class containing $\mathcal{A}$. Since $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$, it follows that $\mathcal{M} \subseteq \mathcal{M}_1$.

Using a similar proof as above, one can show that $\mathcal{M}_2 = \{B \subseteq E : B^c, B \cup A \in \mathcal{M} \text{ for all } A \in \mathcal{M}\}$ is a monotone class over E. Since $\mathcal{M} \subseteq \mathcal{M}_1$, then for each $A \in \mathcal{A}$ one has $A \cup B \in \mathcal{M}$ for all $B \in \mathcal{M}$. This shows that $\mathcal{A} \subseteq \mathcal{M}_2$. Since $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$, it follows that $\mathcal{M} \subseteq \mathcal{M}_2$, and hence $\mathcal{M}$ contains $\emptyset$, is closed under finite unions and complements. Therefore, $\mathcal{M}$ is an algebra.

Proof(d) Parts (b) and (c) imply that $\mathcal{M}$ is a σ-algebra containg $\mathcal{A}$, hence $\sigma(E, \mathcal{A}) \subseteq \mathcal{M}$ (since $\sigma(E, \mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$). On the other hand, $\sigma(E, \mathcal{A})$ is a monotone class containing $\mathcal{A}$ (by part (b)), hence $\mathcal{M} \subseteq \sigma(E, \mathcal{A})$ (since $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$). Therefore, $\mathcal{M} = \sigma(E, \mathcal{A})$.