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Measure and Integration Solutions 8

1. Let $\mathcal{C} = \{(a, \infty) : a \in \mathbb{R}\}$, and let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra over $\mathbb{R}$.

   (a) Let $(E, \mathcal{B})$ be a measurable space. Suppose $f : E \to \mathbb{R}$ satisfies $f^{-1}(C) \in \mathcal{B}$ for all $C \in \mathcal{C}$. Show that f is measurable, i.e. $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}_{\mathbb{R}}$.

   (b) Suppose ν and μ are finite measures on $\mathcal{B}_{\mathbb{R}}$, and $\mu\left(f^{-1}(a, \infty)\right) = \nu\left((a, \infty)\right)$ for all $a \in \mathbb{R}$. Show that $\mu(f^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{B}_{\mathbb{R}}$.

   Proof (a): Since $\mathcal{B}_{\mathbb{R}} = \sigma(\mathbb{R}, \mathcal{C})$, the result follows from Lemma 3.2.1.

   Proof (b): Notice that $\mathcal{C}$ is a π-system generating the Borel σ-algebra, and μ and ν are finite measures agreeing on members of $\mathcal{C}$, thus the result follows from problem 1 exercises 7.

2. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f_n : E \to [-\infty, \infty]$ a sequence of measurable functions. Show that $\sup_n f_n$ and $\inf_n f_n$ are measurable.

   Proof: Let $\overline{\mathbb{R}} = [-\infty, \infty]$. Notice that if $\mathcal{C}_1 = \{[a, \infty] : a \in \overline{\mathbb{R}}\}$ and $\mathcal{C}_2 = \{[-\infty, a] : a \in \overline{\mathbb{R}}\}$, then $\mathcal{B}_{\overline{\mathbb{R}}} = \sigma(\overline{\mathbb{R}}; \mathcal{C}_i)$ for $i = 1, 2$. Hence it suffices to show that $\{\sup_n f_n \leq a\}, \{\inf_n f_n \geq a\} \in \mathcal{B}$ for all $a \in \overline{\mathbb{R}}$. Now,

   $$\{\sup_n f_n \leq a\} = \bigcap_n \{f_n \leq a\} \in \mathcal{B}$$

   and

   $$\{\inf_n f_n \geq a\} = \bigcap_n \{f_n \geq a\} \in \mathcal{B}.$$

3. Let $(E, \mathcal{B}, \mu)$ be a measure space. Suppose $f : E \to [-\infty, \infty]$ is a function such that $f = \sum_{i=1}^n a_i 1_{A_i}$, where $a_1, \cdots, a_n$ are **distinct** elements of $[-\infty, \infty]$ and $A_1, A_2, \cdots, A_n$ are disjoint subsets of E. Show that f is measurable (i.e. $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{B}_{[-\infty,\infty]}$) **if and only if** $A_1, A_2, \cdots, A_n \in \mathcal{B}$.

   Proof: Suppose that f is measurable. Notice that $\{a_i\}$ is closed in $[-\infty, \infty]$, hence $\{a_i\} \in \mathcal{B}_{[-\infty,\infty]}$ for all $i = 1, 2, \cdots, n$. Since f is measurable and $A_1, A_2, \cdots, A_n$ are disjoint, then $A_i = f^{-1}(\{a_i\}) \in \mathcal{B}$.

   Conversely, suppose $A_1, A_2, \cdots, A_n \in \mathcal{B}$, then $1_{A_1}, 1_{A_2}, \cdots 1_{A_n}$ are measurable functions. Hence, $f = \sum_{i=1}^n a_i 1_{A_i}$ is measurable.

4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f : E \to [0, \infty]$ a measurable simple function such that $\int_E f \, d\mu < \infty$. Define $\lambda : \mathcal{B} \to [0, \infty]$ by

   $$\lambda(B) = \int_B f \, d\mu.$$

(a) Show that λ is a **finite** measure on $\mathcal{B}$.

(b) Suppose that $\mu(f = 0) = 0$. Show that $\lambda(B) = 0$ **if and only if** $\mu(B) = 0$.

Proof (a): Since $\int_E f \, d\mu < \infty$, then $\mu(f = \infty) = 0$. Let $a_1, a_2, \cdots, a_m$ be the non-zero distinct finite values of f, then $\int_E f \, d\mu = \sum_{i=1}^m a_i \mu(A_i)$, where $A_i = \{f = a_i\}$. For any $B \in \mathcal{B}$ one has

$$\lambda(B) = \int_E f \cdot 1_B \, d\mu = \sum_{i=1}^m a_i \mu(A_i \cap B).$$

From this, one easily sees that $\lambda(\emptyset) = 0$. Now, suppose $B_1, B_2, \cdots, \in \mathcal{B}$ are pairwise disjoint and let $B = \bigcup_{n=1}^\infty B_n$. Then $1_B = \sum_{n=1}^\infty 1_{B_n}$, and

$$\lambda(B) = \sum_{i=1}^m a_i \mu(A_i \cap B) = \sum_{i=1}^m a_i \sum_{n=1}^\infty \mu(A_i \cap B_n) = \sum_{n=1}^\infty \sum_{i=1}^m a_i \mu(A_i \cap B_n) = \sum_{n=1}^\infty \lambda(B_n).$$

Thus, λ is σ-additive. Since $\lambda(E) = \int_E f \, d\mu < \infty$, it follows that λ is a finite measure on $\mathcal{B}$.

Proof (b): We use the same notation as in the proof of part (a). Suppose $\mu(B) = 0$, then $\mu(A_i \cap B) = 0$ for all $i = 1, 2, \cdots, m$. Hence, $\lambda(B) = \sum_{i=1}^m a_i \mu(A_i \cap B) = 0$. Now, assume that $\lambda(B) = 0$. Since $a_1, a_2, \cdots a_m > 0$ and $\sum_{i=1}^m a_i \mu(A_i \cap B) = 0$, it follows that $\mu(A_i \cap B) = 0$ for all $i = 1, 2, \cdots, m$. Further, since $\mu(f = \infty) = \mu(f = 0) = 0$, then $\mu(E \setminus \bigcup_{i=1}^m A_i) = 0$. Thus, $\mu(B) = \mu(\bigcup_{i=1}^m A_i \cap B) = \sum_{i=1}^m \mu(A_i \cap B) = 0$.