Measure and Integration Solutions 9

1. Consider the measure space $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \lambda_\mathbb{R})$, where $\mathcal{B}_\mathbb{R}$ is the Borel σ-algebra over $\mathbb{R}$ and $\lambda_\mathbb{R}$ is Lebesgue measure on $\mathcal{B}_\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2^{-k} & \text{if } x \in [k, k+1),\ k \in \mathbb{Z},\ k \geq 0. \end{cases}$$

   (a) Show that f is measurable, i.e. $f^{-1}(B) \in \mathcal{B}_\mathbb{R}$ for all $B \in \mathcal{B}_\mathbb{R}$.

   (b) Show that $\int_\mathbb{R} f \, d\lambda_\mathbb{R} = 2$.

   Proof (a): By Lemma 3.2.1 it is enough to show that $f^{-1}((-\infty, a]) \in \mathcal{B}_\mathbb{R}$ for all $a \in \mathbb{R}$. Now,

$$f^{-1}((-\infty, a]) = \begin{cases} \emptyset & \text{if } a < 0 \\ (-\infty, 0] \cup [k+1, \infty) & \text{if } \frac{1}{2^{k+1}} \leq a < \frac{1}{2^k},\ k \geq 0 \\ \mathbb{R} & \text{if } a \geq 1. \end{cases}$$

   In all cases one sees that $f^{-1}((-\infty, a]) \in \mathcal{B}_\mathbb{R}$. Thus, f is measurable.

   Proof (b): Let $\phi_n = f \cdot 1_{(-\infty, n)}$ for $n \geq 1$. Then, $\{\phi_n\}$ is an increasing sequence of non-negative measurable simple functions converging to f. Thus,

$$\begin{aligned} \int_\mathbb{R} f \, d\lambda_\mathbb{R} &= \lim_{n\to\infty} \int_\mathbb{R} \phi_n \, d\lambda_\mathbb{R} \\ &= \lim_{n\to\infty} \sum_{k=0}^{n-1} 2^{-k} \lambda_\mathbb{R}([k, k+1)) \\ &= \sum_{k=0}^{\infty} 2^{-k} = 2. \end{aligned}$$

2. Let $\{f_n\}$ be a sequence in $L^1(\mu)$ such that $\int_E |f_n| d\mu = \alpha$ for all $n \geq 1$, where $\alpha \in (0, \infty)$. Let

$$A_n = \{x : |f_n(x) - \int_E f_n d\mu| \geq n^2\}.$$

   Show that if $\mu(E) = 1$, then $\mu(\limsup_{n\to\infty} A_n) = 0$.

Proof: For each $n \geq 1$, $|f_n(x) - \int_E f_n d\mu|$ is a non-negative measurable function, hence by Markov's inequality (Theorem 3.2.8) one has

$$\mu(A_n) \leq \frac{1}{n^2} \int_E \left(|f_n(x) - \int_E f_n d\mu| \right) d\mu \leq \frac{2\alpha}{n^2}.$$

This shows that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, hence by Borel-Cantelli Lemma one has

$$\mu\left(\limsup_{n\to\infty} A_n\right) = 0.$$

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f, g \in L^1(\mu)$, i.e. $\int_E |f| d\,\mu < \infty$ and $\int_E |g| d\,\mu < \infty$. Show that $\mu(f \neq g) = 0$ **if and only if** $\int_B f\, d\mu = \int_B g\, d\mu$ for all $B \in \mathcal{B}$.

   Proof: Suppose $\mu(f \neq g) = 0$. Since $f \neq g$ if and only if $|f - g| > 0$, then $\mu(|f - g| > 0) = 0$. By Theorem 3.2.8, this implies that $\int_E |f - g| d\mu = 0$. For any $B \in \mathcal{B}$, one has $|f - g| \cdot 1_B \leq |f - g|$ and hence by Lemma 3.2.7

$$\int_B |f - g| d\mu \leq \int_E |f - g| d\mu = 0.$$

   But $|\int_B f d\mu - \int_B g d\mu| \leq \int_B |f - g| d\mu = 0$. Therefore, $\int_B f\, d\mu = \int_B g\, d\mu$ for all $B \in \mathcal{B}$.

   Conversely, suppose $\int_B f\, d\mu = \int_B g\, d\mu$ for all $B \in \mathcal{B}$. Let $A = \{x \in E : f - g \geq 0\}$. Then, $A \in \mathcal{B}$ and $(f - g)^+ = (f - g) \cdot 1_A$. Thus,

$$\int_E (f - g)^+ d\mu = \int_A (f - g) d\mu = 0.$$

   By Theorem 3.2.8, this implies that $\mu((f - g)^+ > 0) = 0$. Similarly, $(f - g)^- = (f - g) \cdot 1_{A^c}$ and

$$\int_E (f - g)^- d\mu = \int_{A^c} (f - g) d\mu = 0,$$

   which implies that $\mu((f - g)^- > 0) = 0$. Therefore,

$$\mu(f \neq g) = \mu(|f - g| > 0) \leq \mu((f - g)^+ > 0) + \mu((f - g)^- > 0) = 0.$$