Solutions Test 1, SCI 113 Spring 2008

- Write your name and student number on each page you hand in.
- You are allowed to use the book Mathematical Techniques by Jordan and Smith and the lecture notes by Frits Beukers.
- You should explain how you have calculated your answers
- Define $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $z \in \mathbb{C}$.

(1) Find **all** solutions of the following system of linear equations

$$\begin{cases} x + y - 3z = 1 \\ 3x - y + 2z = 4 \\ 5x + y - 4z = 6. \end{cases}$$

Solution We write the corresponding augmented matrix

$$\begin{pmatrix} 1 & -1 & -3 & 1 \\ 3 & -1 & 2 & 4 \\ 5 & 1 & -4 & 6 \end{pmatrix},$$

and using Gauss Elimination method we transform it to the matrix

$$\begin{pmatrix} 1 & -1 & -3 & 1 \\ 0 & -4 & 11 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From which we read that the system has infinitely many solutions of the form: $z = t$ is any real number, $y = \dfrac{11}{4}t - \dfrac{1}{4}$ and $x = \dfrac{1}{4}t + \dfrac{5}{4}$. In vector form, the set of all solutions is given by

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4} \begin{pmatrix} t+5 \\ 11t-1 \\ 4t \end{pmatrix}, t \in \mathbb{R} \right\}$$

(2) (a) Given two vectors $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (2, -1, 2)$ in $\mathbb{R}^3$. Compute the angle between vectors $\mathbf{u}$ and $\mathbf{v}$.

(b) Given a point $M = (1, 2)$ and a vector $\mathbf{u} = (3, 5)$ in $\mathbb{R}^2$. Determine the equation of the line passing through M and is perpendicular to $(3, 5)$.

Solution(a) We first calculate $\mathbf{u} \cdot \mathbf{v} = 2 - 1 + 2 = 3$, $|\mathbf{u}| = \sqrt{3}$ and $|\mathbf{v}| = \sqrt{9} = 3$. Hence,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Which implies that $\theta = 54.74°$.

Solution(b) Notice that the equation of a line has the form $ax + by = c$, where $\begin{pmatrix} a \\ b \end{pmatrix}$ is a vector perpendicular to the line. In our case $\mathbf{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is perpendicular to the line, so the line has the equation $3x + 5y = c$. To find c, we plug in the coordinates of the point M in the equation to get $3(1) + 5(2) = c$ so $c = 13$, and the equation is: $3x + 5y = 13$.

(3) (a) Express the following complex numbers in standard form $a+ib$ with $a, b \in R$:
$\dfrac{1+5i}{2-3i}$, $3e^{3-i\pi/3}$, $\pi\cos(\pi i)$.
(b) Express the following numbers in polar (exponential) form $re^{i\theta}$, where $-\pi < \theta \le \pi$:
$(1-i\sqrt{3})^{99}$, $\left(\dfrac{-1+i\sqrt{3}}{1+i}\right)^2$.
(c) Determine all complex solutions of the equation $z^4 = (1-i)^3$.
(d) Use De Moivre's Theorem to find an expression for $\cos^2\theta$ in terms of $\cos(2\theta)$. Use this expression to prove that $\cos(\pi/12) = \dfrac{\sqrt{\sqrt{3}+2}}{2}$.

Solution(a)

$$\frac{1+5i}{2-3i} = \frac{1+5i}{2-3i}\cdot\frac{2+3i}{2+3i} = \frac{-13+13i}{13} = -1+i.$$

$$3e^{3-i\pi/3} = 3e^3\, e^{-i\pi/3} = 3e^3(\cos\pi/3 - i\sin\pi/3) = 3e^3/2 - i\,3e^3\sqrt{3}/2.$$

Solution(b)

$$(1-i\sqrt{3})^{99} = \left(2\,e^{-i\,\pi/3}\right)^{99} = 2^{99}\,e^{-i\,33\pi} = 2^{99}\,e^{-i\,\pi} = 2^{99}\,e^{i\,\pi}.$$

$$\left(\frac{-1+i\sqrt{3}}{1+i}\right)^2 = \frac{-1-i\sqrt{3}}{i} = -\sqrt{3}+i = 2e^{i\,5\pi/6}.$$

Solution(c) First, $z = |z|e^{i\theta}$, and

$$(1-i)^3 = \left(\sqrt{2}e^{-i\,\pi/4}\right)^3 = 2^{3/2}e^{i\,(-3\pi/4+2n\pi)}.$$

Hence

$$|z|^4 e^{i4\theta} = 2^{3/2}e^{i\,(-3\pi/4+2n\pi)}$$

implying that $|z|^4 = 2^{3/2}$ or $|z| = 2^{3/8}$ and $4\theta = -3\pi/4 + 2n\pi$ or $\theta = -3\pi/16 + n\pi/2$, for $n \in \mathbb{N}$. Since the degree of the polynomial is 4, we know that there are only 4 distinct solutions obtained when n is replaced by $0, 1, -1, 2$. Therefore, the solutions are

$$2^{3/8}e^{i\,-3\pi/16},\ 2^{3/8}e^{i\,5\pi/16},\ 2^{3/8}e^{-i\,11\pi/16},\ 2^{3/8}e^{i\,13\pi/16}.$$

Solution From De Moivre's Theorem, we have

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2,$$

leading to

$$\cos 2\theta + i\sin 2\theta = (\cos^2\theta - \sin^2\theta) + i2\sin\theta\cos\theta.$$

Thus,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1,$$

or $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$. Plugging in $\theta = \pi/12$ leads to

$$\cos^2 \pi/12 = \frac{1}{2}(\cos \pi/6 + 1) = \frac{\sqrt{3}+2}{4},$$

giving

$$\cos \pi/12 = \frac{\sqrt{\sqrt{3}+4}}{2}.$$