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Partial Combinatory Algebras of Functions

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Abstract We employ the notions of "sequential function" and "interrogation" (dialogue) in order to define new partial combinatory algebra structures on sets of functions. These structures are analyzed using Longley's preorder-enriched category of partial combinatory algebras and decidable applicative structures. We also investigate total combinatory algebras of partial functions. One of the results is that every realizability topos is a geometric quotient of a realizability topos on a total combinatory algebra.

1 Introduction

Let us think of a computing device which, in the course of its calculations, is allowed to consult an oracle. I wish to keep the intuition of "computing device" as flexible as possible and refrain therefore from a definition, but one requirement I want to stick to is this: the device will use only finitely many oracle queries in any terminating computation (there may be nonterminating computations in which the device just keeps on passing queries to the oracle).

If a terminating computation always results in an output, the device then determines a partial function $\mathcal{O} \stackrel{\Phi}{\to} \mathcal{R}$, where $\mathcal{O}$ is the set of oracles and $\mathcal{R}$ the set of results (outputs). In cases of practical interest, $\mathcal{R}$ is a discrete set such as the set $\mathbb{N}$ of natural numbers, whereas $\mathcal{O}$, like the set of all functions $\mathbb{N} \to \mathbb{N}$, has a nontrivial topology. The finiteness requirement above implies in this example that the partial function $\mathcal{O} \to \mathcal{R}$ is continuous, and this is often taken as the meaning of the Use Principle in Recursion Theory (e.g., [15], p. 50: "The Use Principle asserts that Φ_e is continuous.").

In this paper I concentrate on the situation where $\mathcal{O}$ is the set A^A of functions $A \to A$ for some infinite set A, and queries are of form, "what is your value at $a \in A$?" I argue that here it makes sense to consider a subclass of the class of (partial) continuous functions on A^A: the *sequential* functions. Actually, the computable

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maps Φ will turn out to be sequential (note that the word "sequential" as used here has nothing to do with sequential spaces or sequential continuity, but rather with a certain computation strategy: see Section 2 for a precise definition).

It is shown that the sequential functions play an important role in the construction of *partial combinatory algebras*. I show that Kleene's construction of a partial combinatory algebra structure on $\mathbb{N}^{\mathbb{N}}$ ([5]; see [11], 1.4.3, for a concise exposition) can be generalized to a partial combinatory algebra structure on any set A^A for infinite A.

Then some further analysis is carried out in the case that A itself has a partial combinatory algebra structure and the coding which is necessary for defining the structure of A^A is actually definable from A. A universal property is obtained in Longley's category [6] of partial combinatory algebras and decidable applicative morphisms. We also look at subpartial combinatory algebras of A^A.

Next we discuss partial combinatory algebras on sets of *partial* functions. In the end we obtain a theorem in the theory of realizability toposes: every realizability topos is a geometric quotient of a realizability topos on a total combinatory algebra.

2 Sequential (Partial) Functions

The notion "sequential" has been around for a long time and stems from the study of Plotkin's (and Sazonov's) calculus PCF [12; 13]. For a fairly recent paper discussing various approaches to the matter, see [8].

The notion of a sequential tree was defined in [9]. I slightly modify it here. In order to avoid any ambiguity, I also include a definition of "tree" and the various concepts related to it.

Definition 2.1 A *tree* in this paper is a partially ordered set with a least element (the *root* of the tree) such that for every element x, the set $\{y \mid y \leq x\}$ is a finite linearly ordered subset. An element y is called an *immediate successor* of x if x is the greatest element below y. A *path* through a tree is a maximal linearly ordered subset. A *leaf* of a tree is a maximal element. A tree is *well-founded* if every path through it is finite.

Let A be a set, and T a set of finite functions $p : A' \to A$ with $A' \subset A$. We shall also write $\mathrm{dom}(p)$ for A'. The set T is ordered by inclusion. T is called a *sequential tree* if it is a tree, with the empty function as root, and has the property that for every $p \in T$ which is not a leaf, there is an element $a \notin \mathrm{dom}(p)$ such that for all immediate successors q of p in T, we have $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{a\}$.

A sequential tree is *total* if for each $p \in T$ which is not a leaf, there is $a \notin \mathrm{dom}(p)$ such that the set of immediate successors of p in T is the set of *all* functions q satisfying $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{a\}$.

We shall mainly be interested in total sequential trees. Clearly, for such a tree, every function $f : A \to A$ determines a path through the tree (the set $\{p \in T \mid p \subset f\}$). Suppose F is a function from the set of leaves of T to A. Then F and T determine a partial function $\Phi_{T,F} : A^A \to A$ as follows: $\Phi_{T,F}(\alpha)$ is defined if and only if the path through T determined by α ends in a leaf v, and in that case, $\Phi_{T,F}(\alpha) = F(v)$.

Definition 2.2 A partial function $A^A \to A$ of the form $\Phi_{T,F}$ is called a *partial sequential function*.

Note that a function $\Phi_{T,F}$ is a total function (i.e., everywhere defined) if and only if the tree T is well-founded.

The set A is given the discrete topology and the set of functions A^A the product topology. One of the lemmas underlying the construction of Kleene's $\mathcal{K}_2$ (a partial combinatory algebra of functions $\mathbb{N}^{\mathbb{N}}$) is that when A is countable, every partial continuous function $A^A \to A$ with open domain is partial sequential. In fact, one can replace "open" by "G_δ" in this statement.

If A is uncountable, we still have that every *total* continuous function $A^A \to A$ is sequential, but this may fail for partial functions with open domain, as the following proposition shows.

Proposition 2.3 *Let A be an infinite set. Every continuous function $A^A \to A$ is sequential, but if A is uncountable, there exist partial continuous functions with open domain that are not partial sequential.*

Proof Let $\Phi : A^A \to A$ be continuous. For a finite function $p : A' \to A$ with $A' \subset A$, let $U_p = \{\alpha \in A^A \mid p \subset \alpha\}$ be the open neighborhood determined by p. By continuity, the function Φ has a *base* $\mathcal{B}$, that is, a set of finite functions p such that Φ is constant on U_p and such that for every $\alpha \in A^A$ there is a $p \in \mathcal{B}$ such that $p \subset \alpha$.

Call two finite functions s and t *compatible* if $s \cup t$ is a function (i.e., $s(a) = t(a)$ whenever $a \in \mathrm{dom}(s) \cap \mathrm{dom}(t)$). For arbitrary finite s, let $\mathcal{B}_s$ be the set of those $p \in \mathcal{B}$ such that p and s are compatible.

Claim For any finite function s, either Φ is constant on U_s or there is a finite subset C of $A - \mathrm{dom}(s)$ such that for every $p \in \mathcal{B}_s$, $\mathrm{dom}(p)$ meets C.

Proof of Claim Suppose there is no such C; we will show that Φ is constant on U_s. Given $\alpha, \beta \in U_s$, take $p, q \in \mathcal{B}_s$ such that $p \subset \alpha$, $q \subset \beta$. Since $(\mathrm{dom}(p) \cup \mathrm{dom}(q)) - \mathrm{dom}(s)$ is finite and by assumption not a C as above, we can find $r \in \mathcal{B}_s$ such that $\mathrm{dom}(r) - \mathrm{dom}(s)$ is disjoint from $(\mathrm{dom}(p) \cup \mathrm{dom}(q)) - \mathrm{dom}(s)$. Then r is compatible both with p and with q, and since Φ is constant on U_p, U_q, and U_r, Φ takes the same values on U_p and U_q; hence $\Phi(\alpha) = \Phi(\beta)$. We conclude that Φ is constant on U_s.

We now build a sequential tree for Φ as follows: T will be the union of a sequence $T_0 \subseteq T_1 \subseteq \cdots$ of well-founded sequential trees. Let T_0 consist of only the empty function. Suppose T_n has been defined. For every leaf s of T_n define the set of elements of T_{n+1} extending s as follows: if Φ is constant on U_s, this set is empty (and s is also a leaf of T_{n+1}). Otherwise, pick a finite set C for s as in the Claim, and order C as $\{c_1, \ldots, c_n\}$. Then add for each k, $1 \leq k \leq n$, all functions extending s whose domain is $\mathrm{dom}(s) \cup \{c_1, \ldots, c_k\}$.

Each T_n is clearly a well-founded sequential tree, by induction; and by construction the following is true: if $p \in \mathcal{B}$ and p is compatible with a leaf s of T_n, then either Φ is constant on U_s (in which case s is a leaf of T), or the cardinality of $s \cap p$ is at least n. Hence, since the sets $\{U_p \mid p \in \mathcal{B}\}$ cover A^A, the tree T is well-founded. Define for any leaf s of T, $F(s) = \Phi(\alpha)$ for an arbitrary $\alpha \in U_s$. Then $\Phi = \Phi_{T,F}$.

For the second statement, split A in two disjoint, nonempty subsets A_0, A_1. Let Φ be a partial function which is constant on its domain, and which is only defined on those α for which there is an $a \in A$ with $\alpha(a) \in A_0$. Then Φ cannot be partial sequential. Suppose it is; let T be a sequential tree for it. Choose α such that $\alpha(a) \notin A_0$ for each $a \in A$. Then the path through T determined by α must be infinite by assumption on T, but the union of this path is a partial function on A with *countable* domain A'. But then any function α' which agrees with α on A' but has $\alpha'(a) \in A_0$ for some $a \notin A'$ will determine the same infinite path, although $\Phi(\alpha')$ should be defined. □

Partial sequential functions $A^A \to A$ can be coded by elements of A^A in the following way. Let A^* be the free monoid on A, that is, the set of finite sequences of elements of A. Fix an injective function

$$(a_0, \ldots, a_{n-1}) \mapsto \langle a_0, \ldots, a_{n-1} \rangle : A^* \to A.$$

The elements in the image of this map are called *coded sequences*. Let q and r (for "query" and "result") be two specified, distinct elements of A. With these data we define a partial operation $\varphi : A^A \times A^A \to A$ as follows.

For $\alpha, \beta \in A^A$ and $u = \langle a_0, \ldots, a_{n-1} \rangle$ a coded sequence, call u an *interrogation* of β by α, if for each $j \leq n-1$ there is an $a \in A$ such that $\alpha(\langle a_0, \ldots, a_{j-1} \rangle) = \langle q, a \rangle$ and $\beta(a) = a_j$. Of course, for $j = 0$ this means that $\alpha(\langle\,\rangle) = \langle q, a \rangle$ and $\beta(a) = a_0$. The elements $\langle q, a \rangle$ are called the *queries* of the interrogation.[1]

Note that α and β uniquely determine a sequence of interrogations (the *interrogation process*) which may be finite or infinite. We shall apply the notion of interrogation also to finite functions.

We say that $\varphi(\alpha, \beta)$ *is defined with value* b if there is an interrogation u of β by α such that $\alpha(u) = \langle r, b \rangle$. We call the element $\langle r, b \rangle$ the *result* of the interrogation u.

Write φ_α for the partial function $\beta \mapsto \varphi(\alpha, \beta) : A^A \to A$.

Proposition 2.4 *A partial function $A^A \to A$ is of the form φ_α for some $\alpha \in A^A$, precisely when it is sequential.*

Proof Suppose we have a partial sequential function $\Phi_{T,F}$. Then for any $s \in T$, the sequential tree structure of T induces a linear order on $\mathrm{dom}(s)$, say $\mathrm{dom}(s) = \{c_0, \ldots, c_{n-1}\}$ (and the predecessors of s in T are restrictions of s to subsets $\{c_0, \ldots, c_{j-1}\}$). Let $a_i = s(c_i)$. Let α be any function $A \to A$ such that for each $s \in T$, with c_i, a_i as above, $\alpha(\langle a_0, \ldots, a_{n-1} \rangle) = \langle r, F(s) \rangle$ if s is a leaf of T, and $\alpha(\langle a_0, \ldots, a_{n-1} \rangle) = \langle q, c_n \rangle$ if c_n is the unique element of $\mathrm{dom}(t) - \mathrm{dom}(s)$ for each immediate successor t of s in T. Clearly then, $\varphi_\alpha = \Phi_{T,F}$.

Conversely, given α, let T_0 be the set of all finite functions s such that there exists an interrogation of s by α which contains all the values of s. It is easy to see that T_0 is a total sequential tree. Let us look at the leaves of T_0. If s is such a leaf, we have the following three possibilities:

(1) the interrogation process of s by α is infinite: α continues to ask for information it has already received;
(2) for some interrogation u of s by α, $\alpha(u)$ is neither a query nor a result;
(3) for some interrogation u of s by α, $\alpha(u) = \langle r, b \rangle$ for some b.

Let T result from T_0 by the following: for every leaf s of T_0 for which (1) or (2) holds, choose an injective function $(a_0, a_1, \ldots)$ of $\mathbb{N}$ into $A - \mathrm{dom}(s)$, and add all finite functions of the form $s \cup t$ for which $\mathrm{dom}(t) = \{a_0, \ldots, a_k\}$ for some $k \geq 0$.

Finally, if s is a leaf of T (hence a leaf of T_0 to which (3) applies), define $F(s) = b$ if $\alpha(u) = \langle r, b \rangle$ for the shortest interrogation u of s by α yielding a result. Then $\varphi_\alpha = \Phi_{T,F}$. □

3 A Partial Combinatory Algebra Structure on A^A

In this section, we generalize the definition of Kleene's $\mathcal{K}_2$. Our aim is to prove that with the partial map $\alpha, \beta \mapsto \alpha\beta$ defined below in Definition 3.2, the set A^A has the structure of a *partial combinatory algebra*. Let us first recall what this means.

Definition 3.1 A *partial combinatory algebra* is a set X together with a partial function $X \times X \to X$, written $x, y \mapsto xy$, such that there exist elements k and s in X satisfying the two axioms:

(k) for any $x, y \in X$, kx and $(kx)y$ are defined, and $(kx)y = x$;

(s) for any $x, y, z \in X$, sx and $(sx)y$ are defined, and $((sx)y)z$ is defined precisely if $(xz)(yz)$ is, and these expressions define the same element of X if defined.

The partial function $a, b \mapsto ab$ is called the *application function*.

Definition 3.2 For $\alpha, \beta \in A^A$ and $a \in A$, call $u = \langle a_0, \dots, a_{n-1}\rangle$ an *a-interrogation* of β by α, if for each $j \leq n-1$, there is a $b \in A$ such that $\alpha(\langle a, a_0, \dots, a_{j-1}\rangle) = \langle q, b\rangle$ and $\beta(b) = a_j$. We say that $\varphi^a(\alpha, \beta)$ is defined with value c, if for some a-interrogation u of β by α, $\alpha(u) = \langle r, c\rangle$. Then, define a partial function $A^A \times A^A \to A$, denoted $\alpha\beta \mapsto \alpha\beta$ in the following way: $\alpha\beta$ is defined if and only if for every $a \in A$, $\varphi^a(\alpha, \beta)$ is defined. In that case, $\alpha\beta$ is the function $a \mapsto \varphi^a(\alpha, \beta)$.

The following proposition is straightforward.

Proposition 3.3 *For any α and a, the partial function $\beta \mapsto \varphi^a(\alpha, \beta)$ is sequential. Conversely, suppose that for every $a \in A$ we are given a partial sequential function $F_a : A^A \to A$. Then there is an element α of A^A such that for all $\beta \in A^A$ and all $a \in A$, $\varphi^a(\alpha, \beta)$ is defined if and only if $F_a(\beta)$ is, and equal to it in that case. Hence if, for all a, $F_a(\beta)$ is defined, $\alpha\beta$ is defined and for all a, $\alpha\beta(a) = F_a(\beta)$.*

We shall have to deal with sequential functions of more than one variable.

Definition 3.4 Let T be a set of pairs of finite functions (s, t), ordered by pairwise inclusion. We say that T is a *bisequential tree* if T is a tree, and for any nonleaf $(s, t) \in T$ we have *either* there is $a \notin \mathrm{dom}(s)$ such that the set of immediate successors of (s, t) in T is the set of finite functions (s', t) where s' extends s and $\mathrm{dom}(s') = \mathrm{dom}(s) \cup \{a\}$, *or* there is $b \notin \mathrm{dom}(t)$ such that the set of immediate successors is the set of (s, t') with $\mathrm{dom}(t') = \mathrm{dom}(t) \cup \{b\}$.

Any pair of functions $f, g : A \to A$ determines a unique path through a bisequential tree (the set $\{(s, t) \mid s \subset f, t \subset g\}$), and just as for the case of one variable we say that a partial function $\varphi : A^A \times A^A \to A$ is bisequential if there is a bisequential tree T and a function F from the leaves of T to A such that $\varphi = \Phi_{T,F}$. Here we use the notation $\Phi_{T,F}$ also for functions of two variables in the same way as before.

Lemma 3.5 *Let G be a total bisequential function $A^A \times A^A \to A$. Then there is an element φ_G of A^A such that for all α and β, $\varphi_G\alpha$ is defined, and $\varphi(\varphi_G\alpha, \beta) = G(\alpha, \beta)$.*

Proof Let G be $\Phi_{T,F}$, so T is a bisequential tree, F a function from the leaves of T to A such that $G(\alpha, \beta) = F((p, q))$ for the unique leaf (p, q) determined by T and (α, β). Note that since G is total, the tree T is well-founded.

Call a nonleaf (s, t) of T a $(0, u)$-point if all immediate successors of (s, t) are of form (s', t) with $\mathrm{dom}(s') = \mathrm{dom}(s) \cup \{u\}$; similarly, a $(1, v)$-point has immediate successors (s, t') with $\mathrm{dom}(t') = \mathrm{dom}(t) \cup \{v\}$. Suppose $s(u_0) = a_0, \dots, s(u_{n-1}) = a_{n-1}$, $t(v_0) = b_0, \dots, t(v_{m-1}) = b_{m-1}$ are the values of s and t in the path, in that order. We define the value of φ_G on the interrogation

$$\langle\langle b_0, \dots, b_{m-1}\rangle, a_0, \dots, a_{n-1}\rangle \ = \ \langle\langle \vec{b}\rangle, \vec{a}\rangle.$$

If (s, t) is a $(0, u)$-point, let $\varphi_G(\langle\langle\vec{b}\rangle, \vec{a}\rangle) = \langle q, u\rangle$; if (s, t) is a $(1, v)$-point, let $\varphi_G(\langle\langle\vec{b}\rangle, \vec{a}\rangle) = \langle r, \langle q, v\rangle\rangle$. Finally, if (s, t) is a leaf, $\varphi_G(\langle\langle\vec{b}\rangle, \vec{a}\rangle) = \langle r, \langle r, F(s, t)\rangle$.

It is then straightforward to verify that for every (α, β) passing through the point (s, t), we have for all $j \leq m - 1$ that $(\varphi_G\alpha)(\langle b_0, \dots, b_{j-1}\rangle) = \langle q, v_j\rangle$, and if (s, t) is a leaf of T, we have $(\varphi_G\alpha)(\langle b_0, \dots, b_{m-1}\rangle) = \langle r, G(\alpha, \beta)\rangle$. Since T is well-founded, it is easy to complete the definition of φ_G in such a way that $\varphi_G\alpha$ is always defined. Then also $\varphi(\varphi_G\alpha, \beta) = G(\alpha, \beta)$ as desired. □

Corollary 3.6 *Suppose for each $a \in A$ a total bisequential function $G_a : A^A \times A^A \to A$ is given. Then there is an element φ_G of A^A such that for all $\alpha, \beta \in A^A$ and $a \in A$, $\varphi_G\alpha$ is defined, $(\varphi_G\alpha)\beta$ is defined, and $((\varphi_G\alpha)\beta)(a) = G_a(\alpha, \beta)$.*

Proof Straightforward from Propositions 3.3 and 3.5. □

We can now state the main theorem of this section.

Theorem 3.7 *For an infinite set A, A^A, together with the map $(\alpha, \beta) \mapsto \alpha\beta$, has the structure of a partial combinatory algebra.*

Proof We have to find elements k and s satisfying (k) and (s) of Definition 3.1. Since for any $a \in A$ the map $(\alpha, \beta) \to \alpha(a)$ is bisequential, it follows at once from Corollary 3.6 that there is an element k of A^A such that $(k\alpha)\beta = \alpha$.

For s, we have to do a bit more work. Let α, β be fixed for the moment. We define a function $S^{\alpha\beta}$ as follows: we define recursively the values of $S^{\alpha\beta}$ on elements of the form,

$$\langle a, a_0, \dots, a_{m-1}\rangle,$$

(which we shall also write as $\langle a\rangle * u$, with $u = \langle a_0, \dots, a_{m-1}\rangle$, employing the $*$ notation for concatenation of coded sequences) of which we assume, inductively, that $u = \langle a_0, \dots, a_{m-1}\rangle$ is an a-interrogation of a finite function t by $S^{\alpha\beta}$.

Assume the interrogation u has length n. Determine a maximal sequence,

$$(v_0^0, \dots, v_0^{n_0-1}, b_0, w_0^0, \dots, w_0^{m_0-1}, c_0, \dots, v_j^0, \dots, v_j^{n_j-1}, \\ b_j, w_j^0, \dots, w_j^{m_j-1}, c_j, \dots),$$

of length $\leq n$ such that for any segment

$$(v_j^0, \dots, v_j^{n_j-1}, b_j, w_j^0, \dots, w_j^{m_j-1}, c_j)$$

or initial parts of it, the following hold (where applicable):

(i) $\langle v_j^0, \dots, v_j^{n_j-1}\rangle$ is an $\langle a, c_0, \dots, c_{j-1}\rangle$-interrogation of t by α with result $\langle q, b_j\rangle$ (so the value of α on this sequence is $\langle r, \langle q, b_j\rangle\rangle$);

(ii) $\langle w_j^0, \dots, w_j^{m_j-1}\rangle$ is a b_j-interrogation of t by β with result c_j.

This means that for each j and each $k \le n_j - 1$ there is a d such that

$$\alpha(\langle\langle a, c_0, \dots, c_{j-1}\rangle, v_j^0, \dots, v_j^{k-1}\rangle) = \langle q, d\rangle$$

and $t(d) = v_j^k$, and similarly for each j and each $k \le m_j - 1$ there is an e such that

$$\beta(\langle b_j, w_j^0, \dots, w_j^{k-1}\rangle) = \langle q, e\rangle$$

and $t(e) = w_j^k$.

We define the value $S^{\alpha\beta}(\langle a\rangle * u)$ as follows:

(1) if the sequence ends in $(v_j^0, \dots, v_j^k)$ and

$$\alpha(\langle\langle a, c_0, \dots, c_{j-1}\rangle, v_j^0, \dots, v_j^k\rangle) \;=\; \langle q, x\rangle$$

then $S^{\alpha\beta}(\langle a\rangle * u) = \langle q, x\rangle$;

(2) if the sequence ends in $(b_j, w_j^0, \dots, w_j^k)$ and

$$\beta(\langle b_j w_j^0, \dots, w_j^k\rangle) \;=\; \langle q, x\rangle$$

then $S^{\alpha\beta}(\langle a\rangle * u) = \langle q, x\rangle$;

(3) if the sequence ends in $(v_j^0, \dots, v_j^{n_j-1})$ and

$$\alpha(\langle\langle a, c_0, \dots, c_{j-1}\rangle, v_j^0, \dots, v_j^{n_j-1}\rangle) \;=\; \langle r, \langle r, y\rangle\rangle$$

then $S^{\alpha\beta}(\langle a\rangle * u) = \langle r, y\rangle$;

(4) in all other cases, $\alpha(\langle a\rangle * u) = \langle q, q\rangle$.

Now it is a matter of straightforward verification that if γ is any function extending t, and $\alpha\gamma$, $\beta\gamma$ are defined, then the sequence $c_0, \dots, c_j$ forms an a-interrogation of $\beta\gamma$ by $\alpha\gamma$. Hence, if $\alpha\gamma$ and $\beta\gamma$ are both defined, $(S^{\alpha\beta}\gamma)(a)$ is defined precisely when $((\alpha\gamma)(\beta\gamma))(a)$ is, and equal to it in that case.

It is also left to the reader to check by inspection of the definition of $S^{\alpha\beta}$ that for fixed a and u, the function

$$(\alpha, \beta) \mapsto S^{\alpha\beta}(\langle a\rangle * u)$$

is bisequential. By Corollary 3.6 it follows that there is an element $\sigma \in A^A$ such that for all α and β, $\sigma\alpha$ and $(\sigma\alpha)\beta$ are defined, and $(\sigma\alpha)\beta = S^{\alpha\beta}$.

One is now tempted to say, "Then by the remarks above, this σ satisfies axiom (s) of Definition 3.1. We conclude that A^A, with the given partial map $(\alpha, \beta) \mapsto \alpha\beta$, is a partial combinatory algebra, as claimed." But actually there is a snag, as was pointed out to me by the second referee: certainly, if $(\alpha\gamma)(\beta\gamma)$ is defined then so is $((\sigma\alpha)\beta)\gamma$, and equal to it; and if $\alpha\gamma$, $\beta\gamma$ and $((\sigma\alpha)\beta)\gamma$ are defined, then so is $(\alpha\gamma)(\beta\gamma)$ and it is equal to $((\sigma\alpha)\beta)\gamma$. But $((\sigma\alpha)\beta)\gamma$ may be defined while $\alpha\gamma$ or $\beta\gamma$ are not.[2]

In order to remedy this, we modify the definition of $S^{\alpha\beta}$ as follows. We define a function $\Sigma^{\alpha\beta}$ by saying what its a-interrogations are on a function t: these are of form v, $v*w$, or $v*w*z$ where v is an a-interrogation of t by α; if v has a result, then w is an a-interrogation of t by β, and if w has a result, then z is an a-interrogation of t by $S^{\alpha\beta}$. If z has a result, so $S^{\alpha\beta}(\langle a\rangle * z) = \langle r, b\rangle$, then $\Sigma^{\alpha\beta}(\langle a\rangle * v * w * z) = \langle r, b\rangle$.

This means that $\varphi^a(\Sigma^{\alpha\beta}, \gamma)$ is defined if and only if $\varphi^a(\alpha, \gamma)$, $\varphi^a(\beta, \gamma)$, and $\varphi^a(S^{\alpha\beta}, \gamma)$ are defined, and if this is the case, then $\varphi^a(\Sigma^{\alpha\beta}, \gamma) = \varphi^a(S^{\alpha\beta}, \gamma)$. It follows that if $\Sigma^{\alpha\beta}\gamma$ is defined, we have that $\alpha\gamma$ and $\beta\gamma$ are defined and that $\Sigma^{\alpha\beta}\gamma = (\alpha\gamma)(\beta\gamma)$.

Just as before, one checks by inspection that for fixed a and u, the function

$$(\alpha,\beta)\mapsto \Sigma^{\alpha\beta}(\langle a\rangle * u)$$

is bisequential, so that by Corollary 3.6 there is an $s\in A^A$ such that for each α and β, $(s\alpha)\beta=\Sigma^{\alpha\beta}$. This s then does satisfy axiom (s) of Definition 3.1, and we can now legitimately conclude that A^A is a partial combinatory algebra, as desired. □

We shall denote the partial combinatory algebra on A^A by $\mathcal{K}_2(A)$.

4 Further Analysis of $\mathcal{K}_2(A)$

In this section we try to analyze the construction of $\mathcal{K}_2(A)$ a bit, from the point of view of Longley's 2-category PCA of partial combinatory algebras (first defined in [6]; there is also a description in [11]).

Convention From now on, when dealing with iterated applications we shall use the familiar convention of "associating to the left"; that is, we write $abcd$ instead of $((ab)c)d$.

PCA is a preorder-enriched category. The objects are partial combinatory algebras. Given two such, A and B, a 1-cell, or *applicative morphism*, from A to B is a total relation γ from A to B (we think of γ as a function $A\to\mathcal{P}^*(B)$ into the set of nonempty subsets of B), with the property that there exists an element $r\in B$ such that, whenever $a,a'\in A$, $b\in\gamma(a)$, $b'\in\gamma(a')$ and aa' is defined, then rbb' is defined and an element of $\gamma(aa')$. The element r is called a *realizer* for γ.

If $\gamma,\gamma':A\to B$ are two applicative morphisms, we say $\gamma\preceq\gamma'$ if there is an element $s\in B$ such that for all $a\in A$ and all $b\in\gamma(a)$, sb is defined and an element of $\gamma'(a)$. The element s is said to *realize* $\gamma\preceq\gamma'$. For two parallel arrows $\gamma,\gamma':A\to B$ we write $\gamma\cong\gamma'$ if $\gamma\preceq\gamma'$ and $\gamma'\preceq\gamma$.

It is part of the theory of partial combinatory algebras that every partial combinatory algebra A contains elements $\bot$, $\top$, and C (thought of as "Booleans" and "definition by cases"), satisfying for all $a,b\in A$:

$$C\top ab=a \quad\text{and}\quad C\bot ab=b.$$

Instead of $Cvab$ we write "If v then a else b."

Suppose $\gamma:A\to B$ is an applicative morphism and $\top_A,\bot_A$ are Booleans in A, $\top_B,\bot_B$ are Booleans in B. We call the morphism γ *decidable* if there is an element $d\in B$ (a *decider* for γ) such that for all $b\in\gamma(\top_A)$, $db=\top_B$ and for all $b\in\gamma(\bot_A)$, $db=\bot_B$. There is a subcategory of PCA on the decidable applicative morphisms.

One further definition: if $\gamma:A\to B$ is an applicative morphism and f is a partial function $A\to A$, then f is said to be *representable* with respect to γ, if there is an element $r_f\in B$ (which then *represents* f), such that for all $a\in\mathrm{dom}(f)$ and all $b\in\gamma(a)$, r_fb is defined and an element of $\gamma(f(a))$. We shall just say "f is representable" if we mean that f is representable with respect to the identity morphism on A.

Proposition 4.1 *For $a\in A$ let $\hat{a}$ denote the constant function with value a. For any partial combinatory algebra structure on A, the map $\gamma(a)=\{\hat{a}\}$ defines a decidable applicative morphism $A\to\mathcal{K}_2(A)$. Every total function $A\to A$ is representable with respect to γ.*

Proof This is easy. Let ρ be any element of A^A satisfying

$$\begin{array}{rcl} \rho(\langle\langle x\rangle\rangle) & = & \langle r, \langle q, q\rangle\rangle \\ \rho(\langle\langle x, b\rangle\rangle) & = & \langle q, q\rangle \\ \rho(\langle\langle x, b\rangle, a\rangle) & = & \begin{cases} \langle r, \langle r, ab\rangle\rangle & \text{if } ab \text{ is defined in } A \\ \langle r, \langle q, q\rangle\rangle & \text{otherwise} \end{cases} \\ \rho(\langle n\rangle) & = & \langle r, r\rangle \text{ if } n \neq \langle x\rangle \text{ and } n \neq \langle x, b\rangle\,. \end{array}$$

Then it is easily verified that if ab is defined in A, $\rho\hat{a}$ is defined and $\rho\hat{a}\hat{b} = \widehat{ab}$. Hence, γ is an applicative morphism. Furthermore, for any good choice of Booleans $\top, \bot$ in A one can take $\hat{\top}, \hat{\bot}$ for Booleans in A^A, so decidability is easy. That every $f : A \to A$ is representable is left to the reader. $\square$

At this point I wish to collect a few bits of notation and theory of partial combinatory algebras; everything can be found in Sections 1.1 and 1.3 of [11]. Let A be a partial combinatory algebra.

(1) If t and s are terms built up from elements of A and the application function, $t\downarrow$ means "t is defined," and $t \simeq s$ means t is defined if and only if s is, and they denote the same element of A if defined.

(2) Given a term $t(x_1, \ldots, x_{n+1})$ built up from elements of A, variables $x_1, \ldots, x_{n+1}$ and the application function, there is a standard construction for an element $\langle x_1 \cdots x_{n+1}\rangle t$ of A which satisfies

$$\begin{array}{l} (\langle x_1 \cdots x_{n+1}\rangle t)a_1 \cdots a_n\downarrow \\ (\langle x_1 \cdots x_{n+1}\rangle t)a_1 \cdots a_{n+1} \simeq t(a_1, \ldots, a_{n+1})\,. \end{array}$$

The reader should keep in mind that the notation $\langle x_1 \cdots x_{n+1}\rangle t$ has nothing to do with the notation $\langle x_1, \ldots, x_{n+1}\rangle$ we have been using for coded sequences.

(3) A has elements p, p_0, p_1 (*pairing* and *unpairing* combinators) such that $pab\downarrow$, $p_0(pab) = a$, and $p_1(pab) = b$; A contains a copy $\{\bar{0}, \bar{1}, \ldots\}$ of the natural numbers such that any computable function on the natural numbers is represented by an element of A, and A has a standard coding of tuples $[\,\cdot, \ldots, \cdot\,]$ together with elements representing the basic manipulation of these.

(4) A has a *fixed-point operator*: an element z such that for all $f, x \in A$: $zf\downarrow$ and $zfx \simeq f(zf)x$ (this is also referred to as "the recursion theorem in A").

It is clear that the construction of $\mathcal{K}_2(A)$ given above depends on the coding of tuples $\langle\,\cdot, \ldots, \cdot\,\rangle$ and the elements q and r. Since we wish to study the connection to A in the case A itself has the structure of a partial combinatory algebra, we make the following definition.

Definition 4.2 Suppose A is an infinite set and $(A, \cdot)$ a partial combinatory algebra structure on A. We say that $\mathcal{K}_2(A)$ is *based on* $(A, \cdot)$ if, in the definition of an interrogation of β by α, we have used the standard coding $[\,\cdot, \ldots, \cdot\,]$ of A, q and r are, respectively, the Booleans $\bot$ and $\top$, and the values of α at such interrogations are $p\bot u$ or $p\top u$.

We say that $\mathcal{K}_2(A)$ is *compatible with* $(A, \cdot)$ if there are elements $a, b, c \in A$ such that

(i) for every tuple $u_0, \ldots, u_{n-1}$, $a(\langle u_0, \ldots, u_{n-1}\rangle) = [u_0, \ldots, u_{n-1}]$ and $b([u_0, \ldots, u_{n-1}]) = \langle u_0, \ldots, u_{n-1}\rangle$;

(ii) $cq = \bot$ and $cr = \top$.

Theorem 4.3 *Suppose $(A, \cdot)$ is a partial combinatory algebra and $\mathcal{K}_2(A)$ is based on $(A, \cdot)$. Let $\gamma : A \to \mathcal{K}_2(A)$ be the applicative morphism of Proposition 4.1. Then for any decidable applicative morphism $\delta : A \to B$ such that every total function $A \to A$ is representable with respect to δ, there is a greatest decidable applicative morphism $\varepsilon : \mathcal{K}_2(A) \to B$ such that $\varepsilon\gamma \cong \delta$. Here, "greatest" and "$\cong$" refer to the preorder on applicative morphisms.*

Proof Given δ, define ε as follows: $\varepsilon(\alpha)$ is the (nonempty) set of elements $b \in B$ which represent α with respect to δ. First we show that ε is an applicative morphism: we have to construct a realizer for ε. The proof below may appear a bit technical. However, the reader should bear in mind that what we are doing is coding up the "interrogation-type" application of $\mathcal{K}_2(A)$ within B.

Let r be a realizer for δ and d a decider for δ. Let p, p_0, p_1 in A be the pairing and unpairing combinators, and $[\cdot, \dots, \cdot]$ the standard coding of tuples in A. Choose $\pi \in \delta(p)$ and $\pi_i \in \delta(p_i)$, for $i = 0, 1$. Let c be an element of B such that, if $u = [u_0, \dots, u_{k-1}]$ in A and $y \in A$, $v \in \delta(u)$, $x \in \delta(y)$, then $cvx \in \delta([u_0, \dots, u_{k-1}, y])$. Let $s \in B$ be such that if $y \in A$ and $x \in \delta(y)$, then $sx \in \delta([y])$.

Using the fixed-point combinator in B, find $F \in B$ satisfying for all $a, b, v \in B$: $Fab\downarrow$ and

$$Fabv \simeq \begin{array}{l} \text{If } d(r\pi_0(av)) \text{ then } r\pi_1(av) \text{ else} \\ Fab(cv(rb(r\pi_1(av)))) \,. \end{array}$$

Now suppose $a \in \varepsilon(\alpha)$, $b \in \varepsilon(\beta)$.

Claim For any $y \in A$, $x \in \delta(y)$, and any y-interrogation $u = [u_0, \dots, u_{k-1}]$ of β by α, there is a $v \in \delta([y, u_0, \dots, u_{k-1}])$ such that $Fab(sx) \simeq Fabv$.

This claim is proved by induction on k. For $k = 0$, since $sx \in \delta([y])$ there is nothing to prove.

Suppose the Claim holds for $j \leq k$, and $[u_0, \dots, u_k]$ is a y-interrogation. By induction hypothesis there is a $v \in \delta([y, u_0, \dots, u_{k-1}])$ such that $Fab(sx) \simeq Fabv$. Since $[u_0, \dots, u_k]$ is a y-interrogation of β by α, we have $\alpha([y, u_0, \dots, u_{k-1}]) = p\bot e$ and $\beta(e) = u_k$, for some $e \in A$. Since $av \in \delta(\alpha([y, u_0, \dots, u_{k-1}])) = \delta(p\bot e)$ we have $r\pi_0(av) \in \delta(\bot)$ so $d(r\pi_0(av)) = \bot$ in B. By definition of F, we have

$$Fab(sx) \simeq Fabv \simeq Fab(cv(rb(r\pi_1(av)))).$$

It is easily checked that $cv(rb(r\pi_1(av)))$ is an element of $\delta([y, u_0, \dots, u_k])$. This proves the Claim.

Now if $u = [u_0, \dots, u_{k-1}]$ is a y-interrogation of β by α with result g, that is to say $\alpha([y, u_0, \dots, u_{k-1}]) = p\top g$, and $v \in \delta([y, u_0, \dots, u_{k-1}])$ is as in the Claim, then by definition of F,

$$Fabv = r\pi_1(av) \in \delta(g).$$

We conclude that if $\alpha\beta(y) = g$ then $Fab(sx) \in \delta(g)$; hence, if $\alpha\beta\downarrow$, then $\langle x\rangle Fab(sx) \in \varepsilon(\alpha\beta)$. Therefore, the element $\rho = \langle abx\rangle Fab(sx)$ is a realizer for ε.

That ε is decidable follows easily from the fact that δ is, and the fact that in $\mathcal{K}_2(A)$ we may take $\hat{\bot}$ and $\hat{\top}$ for the Booleans. If $b \in B$ is an element of $\varepsilon\gamma(a)$, that is, b represents $\hat{a}$ with respect to δ, then for any chosen, fixed $\xi \in \bigcup_{a' \in A} \delta(a')$ we have $b\xi \in \delta(a)$ so $\langle b\rangle b\xi$ realizes $\varepsilon\gamma \preceq \delta$; conversely if $b \in \delta(a)$ then the element $\langle x\rangle b \in B$ clearly represents $\hat{a}$ with respect to δ. So we see that $\varepsilon\gamma \cong \delta$.

In order to see that ε is the *greatest* applicative morphism satisfying $\varepsilon\gamma \cong \delta$, suppose ε' is another one. Suppose r' realizes ε', s realizes that $\varepsilon'\gamma \preceq \delta$, and t realizes that $\delta \preceq \varepsilon\gamma$. In $\mathcal{K}_2(A)$ there is an element σ such that for all $\alpha \in A^A$ and $a \in A$, $\sigma\alpha\hat{a} = \widehat{\alpha(a)}$ (this is left to the reader). Choose $\tau \in \varepsilon'(\sigma)$.

Let $\alpha \in A^A$ and $a \in A$ be arbitrary. Suppose $z \in \varepsilon'(\alpha)$. If $x \in \delta(a)$ then $tx \in \varepsilon'(\gamma(a)) = \varepsilon'(\hat{a})$, so $r'(r'\tau z)(tx) \in \varepsilon'(\widehat{\alpha(a)}) = \varepsilon'(\gamma(\alpha(a)))$; hence $s(r'(r'\tau z)(tx)) \in \delta(\alpha(a))$. We conclude that $\langle x\rangle s(r'(r'\tau z)(tx))$ represents α with respect to δ, in other words, is an element of $\varepsilon(\alpha)$. Therefore, $\langle zx\rangle s(r'(r'\tau z)(tx))$ realizes $\varepsilon' \preceq \varepsilon$, as was to be proved. $\square$

Of course, Theorem 4.3 also works if $\mathcal{K}_2(A)$ is compatible with A.

4.1 Sub-pcas of $\mathcal{K}_2(A)$ We now turn our attention to sub-partial combinatory algebras of $\mathcal{K}_2(A)$: subsets $B \subset A^A$ such that, whenever $\alpha, \beta \in B$ and $\alpha\beta\downarrow$ in $\mathcal{K}_2(A)$, then $\alpha\beta \in B$, and, moreover, B with the inherited partial application function is a partial combinatory algebra. For brevity, let's call such a B a sub-pca of $\mathcal{K}_2(A)$.

A stronger notion, which is relevant to relative realizability (see [4; 3]), requires B to contain elements k and s which satisfy the axioms (k) and (s) of Definition 3.1 both with respect to B and with respect to $\mathcal{K}_2$. We call such sub-pcas *elementary*. Examples of elementary sub-pcas are the inclusion of Rec in $\mathcal{K}_2$, where Rec is the set of total recursive functions, or $\Delta_n \subset \mathcal{K}_2$, or RE $\subset \mathcal{P}(\omega)$, where RE is the set of recursively enumerable subsets of $\mathbb{N}$.

In the case that A has a partial combinatory algebra structure and $\mathcal{K}_2(A)$ is compatible with A, we have an instrument for studying sub-pcas of $\mathcal{K}_2(A)$: the preorder $\leq_T$ on partial functions $A \to A$, defined in [10]. There, the following theorem is proved.

Theorem 4.4 *Let A be a partial combinatory algebra and $f : A \to A$ a partial function. There is a partial combinatory algebra $A[f]$ and a decidable applicative morphism $\iota_f : A \to A[f]$ such that f is representable with respect to ι_f, and, moreover, any decidable applicative morphism $\gamma : A \to B$ such that f is representable with respect to γ, factors uniquely through ι_f.*

One can then define, for two partial functions $f, g : A \to A$: $f \leq_T g$ if f is representable with respect to ι_g. This gives a preorder on the set of partial endofunctions on A, which in the case that A is $\mathcal{K}_1$ (the partial combinatory algebra of indices of partial recursive functions) and f and g are total functions, coincides with Turing reducibility.

Moreover, $A[f]$ is defined as follows. The underlying set is A itself, and one defines a "b-interrogation of f by a" just as in the definition of $\mathcal{K}_2(A)$ above, but now using application in A; that is, it is a coded sequence $u = [u_0, \ldots, u_{n-1}]$ such that for each $j < n$ there is a $v \in A$ such that $a([b, u_0, \ldots, u_{j-1}]) = [\bot, v]$ and $f(v) = u_j$. Then $a \cdot^f b = c$ if there is a b-interrogation $u = [u_0, \ldots, u_{n-1}]$ of f by a such that $a([b, u_0, \ldots, u_{n-1}]) = [\top, c]$. The partial map $a, b \mapsto a \cdot^f b$ is the application function for $A[f]$.

We see that if in $\mathcal{K}_2(A)$ the element α is representable in A, by $a \in A$, and $\alpha\beta$ is defined, then $\alpha\beta(x) = a \cdot^\beta x$ for every $x \in A$. We see that $\alpha\beta$ is representable in $A[\beta]$, so $\alpha\beta \leq_T \beta$. We are led to conjecture that a sub-pca of $\mathcal{K}_2(A)$ should be *downward closed* with respect to the preorder $\leq_T$. Let us see what can be said about this.

Proposition 4.5 *Let A be a partial combinatory algebra and suppose $\mathcal{K}_2(A)$ is compatible with A. Suppose $B \subset A^A$ is nonempty and closed under the application function (if $\alpha, \beta \in B$ and $\alpha\beta\downarrow$, then $\alpha\beta \in B$).*

(i) *if for every $a \in A$ there is an element $\alpha \in B$ which extends the partial function represented by a, then B is downward closed with respect to $\leq_T$;*
(ii) *without the hypothesis of* (i)*, the result may fail, even if B is a sub-pca of $\mathcal{K}_2(A)$.*

Proof (i) Suppose $\gamma \in B$ and $\beta \leq_T \gamma$. Then there is an $a \in A$ such that for all $x \in A$, $a \cdot^{\gamma} x = \beta(x)$. If $\alpha \in B$ extends the partial function represented by a, we have $\alpha\gamma = \beta$. So $\beta \in B$, and B is downward closed with respect to $\leq_T$.

(ii) My counterexample is a (nonelementary) sub-pca of Kleene's original $\mathcal{K}_2$. It is easiest to formulate in the original definition of $\mathcal{K}_2$: for $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ we say $\alpha\beta(x) = y$ if there is an n such that

$$\begin{array}{rcll} \alpha\langle x, \beta(0), \ldots, \beta(n-1)\rangle & = & y+1 & \\ \alpha\langle x, \beta(0), \ldots, \beta(j-1)\rangle & = & 0 & \text{for all } j < n\,. \end{array}$$

Then $\alpha\beta$ is defined if for all x there is a y such that $\alpha\beta(x) = y$.

Now define $E_0 = \langle\,\rangle$; $E_{n+1} = \langle E_n\rangle$. Let $B \subset \mathbb{N}^{\mathbb{N}}$ be given by

$$B \;=\; \{\alpha \in \mathbb{N}^{\mathbb{N}} \mid \text{for all } n, \alpha(E_n) = n\}\,.$$

It is easy to check that if $\alpha \in B$ and $\alpha\beta$ is defined in $\mathcal{K}_2$, then $\alpha\beta \in B$. Moreover, if k' and s' are the functions in B which agree with k (and s, respectively) outside $\{E_0, E_1, \ldots\}$, then k' and s' satisfy the axioms (k) and (s) of Definition 3.1. So B is a sub-pca of $\mathcal{K}_2$, but evidently not closed under "recursive in." □

Remark 4.6 The result of part (i) in Proposition 4.5 above can be strengthened a bit. In ordinary recursion theory, the poset of Turing degrees is a join-semilattice. It is not clear whether this is so for the general notion of $\leq_T$ considered here (but see Section 5), but one can define the following: for $\alpha, \gamma_1, \ldots, \gamma_n \in A^A$ say $\alpha \leq_T (\gamma_1, \ldots, \gamma_n)$ if α is representable in $A[\gamma_1, \ldots, \gamma_n]$. Call $B \subset A^A$ an *ideal* if whenever $\gamma_1, \ldots, \gamma_n \in B$ and $\alpha \leq_T (\gamma_1, \ldots, \gamma_n)$, then $\alpha \in B$. One can prove that if B satisfies the hypothesis of (i), then B is an ideal.

Remark 4.7 About part (ii): I do not know whether there exist elementary sub-pcas of $\mathcal{K}_2$ that are not downward closed with respect to $\leq_T$.

Remark 4.8 I would have liked to include a statement in Proposition 4.5 saying that if B is downward closed with respect to $\leq_T$, then B is an elementary sub-pca of $\mathcal{K}_2(A)$. The intuitive reason being that k and s are definable in A, hence, $\leq_T \beta$ for every $\beta \in B$, hence, in B. However, this fails in general, because of the need of making k and s total. We have had to define k and s also outside the relevant interrogations. But in a general partial combinatory algebra it is not decidable whether or not a given element is a pair, or a coded sequence. We shall see that this problem disappears when we consider partial combinatory algebras of partial functions in Section 5.

Theorem 4.3 can be generalized to certain sub-pcas of $\mathcal{K}_2(A)$. The proof is straightforward.

Proposition 4.9 *Suppose $B \subset A^A$ is a sub-pca which contains all constant functions $\hat{a}$ for $a \in A$ and the function ρ from the proof of Proposition 4.1. Then there is a decidable applicative morphism $\gamma : A \to B$ which has the property that whenever $\delta : A \to C$ is decidable and every element of B is representable with respect to δ, then there is $\varepsilon : B \to C$ such that $\varepsilon\gamma \cong \delta$. If, moreover, B contains an element σ such that for all $\alpha \in B$ and $a \in A$, $\sigma\alpha\hat{a} = \widehat{\alpha(a)}$, then ε is greatest with this property.*

5 Total Combinatory Algebras of Partial Functions

With some care, the whole set-up of this paper generalizes to the set $\mathrm{Ptl}(A, A)$ of partial functions $A \to A$. For each $\alpha \in \mathrm{Ptl}(A, A)$ we have a partial function $\varphi_\alpha : \mathrm{Ptl}(A, A) \to A$ given by interrogations. Modifying the definition of sequential functions in such a way that we now consider nontotal sequential trees T and *partial* maps F from the leaves of T to A, we easily see that a partial function $\mathrm{Ptl}(A, A) \to A$ is sequential, precisely if it is of the form φ_α for some $\alpha \in \mathrm{Ptl}(A, A)$. Note that in this case, given φ_α, we can (much more simply than in the proof of Proposition 2.4) define the corresponding sequential tree as the set of those finite functions s such that there is an interrogation of s by α which contains all values of s and is in the domain of α. Finally, for a leaf s of the tree we can define $F(s) = b$ if there is an interrogation u of s by α such that $\alpha(u) = \langle r, b\rangle$.

Quite similarly to Section 3, we have a partial combinatory algebra structure on $\mathrm{Ptl}(A, A)$. This generalizes the construction of $\mathcal{B}$ (for $A = \mathbb{N}$) in [9; 7]. Just as in Definition 3.2 we have a partial function $\varphi^a : \mathrm{Ptl}(A, A) \times \mathrm{Ptl}(A, A) \to A$ for each $a \in A$, and hence a total function $\alpha, \beta \mapsto \alpha\beta : \mathrm{Ptl}(A, A) \times \mathrm{Ptl}(A, A) \to \mathrm{Ptl}(A, A)$. The notion of a bisequential function is also straightforward, and analogously to Lemma 3.5 we have, for every *partial* bisequential function $G : \mathrm{Ptl}(A, A) \times \mathrm{Ptl}(A, A) \to A$, an element φ_G such that for all α and β, $\varphi_{\varphi_G\alpha}(\beta) \simeq G(\alpha, \beta)$. Again, this is simpler than in the case of total functions: no artificial construction in order to make sure that $\varphi_G\alpha$ is a total function is required.

The proof of Theorem 3.7 also simplifies, because the elements k and s need not be artificially extended beyond what they have to perform on the relevant interrogations. It follows that if the coding apparatus needed for the application on $\mathrm{Ptl}(A, A)$ is the standard coding of A, k and s can be chosen to be representable in A.

Let us write $\mathcal{B}(A)$ for the partial combinatory algebra structure on $\mathrm{Ptl}(A, A)$.[3] Since the application function is total, we speak of a *(total) combinatory algebra*. We shall say that $\mathcal{B}(A)$ *is compatible with* A *(based on* A*)* if the obvious analogue of Definition 4.2 holds.

We have the same map $\gamma : A \to \mathcal{B}(A)$ as in Proposition 4.1; it is decidable, and every function $A \to A$ is representable with respect to γ. It is worth noting that in the partial case, the definition of ρ in the proof of Proposition 4.1 can be simplified: we simply define

$$\rho(\langle\langle x, b\rangle, a\rangle) \;\simeq\; \langle r, \langle r, ab\rangle\rangle$$

and don't need to define ρ outside the set of elements of form $\langle\langle x\rangle\rangle$ or $\langle\langle x, b\rangle\rangle$. It follows that if $\mathcal{B}(A)$ is compatible with A, this function ρ is representable in A. The combinatory algebra $\mathcal{B}(A)$ satisfies a similar semi-universal property as the one given for $\mathcal{K}_2(A)$ in Theorem 4.3, with the map $\gamma : A \to \mathcal{B}(A)$, provided $\mathcal{B}(A)$ is compatible with A.

Proposition 5.1 *Suppose A is a partial combinatory algebra and $\mathcal{B}(A)$ is compatible with A. For every decidable applicative morphism $A \xrightarrow{\delta} B$ which has the property that every partial function $A \to A$ is representable with respect to δ, there is a greatest decidable applicative morphism $\varepsilon : \mathcal{B}(A) \to B$ such that $\varepsilon\gamma \cong \delta$.*

Corollary 5.2 *$\mathcal{K}_2(A)$ is an elementary sub-pca of $\mathcal{B}(A)$ and a retract of it in the category of partial combinatory algebras and isomorphism classes of applicative morphisms.*

Proof The choice of k and s we made for $\mathcal{K}_2(A)$ also works for $\mathcal{B}(A)$: so the inclusion $i : A^A \to \mathrm{Ptl}(A, A)$ is elementary; it is also an applicative morphism, and decidable. Furthermore, if we apply the semi-universal property of $\mathcal{B}(A)$ to the diagram

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \mathcal{K}_2(A) \\ \mathcal{B}(A) & \end{tikzcd}$$

we obtain an applicative map $\varepsilon : \mathcal{B}(A) \to \mathcal{K}_2(A)$. Concretely,

$$\varepsilon(\alpha) \;=\; \{\beta \mid \text{for all } a \in \mathrm{dom}(\alpha), \beta\hat{a} = \widehat{\alpha(a)}\}.$$

It is not hard to show that εi is isomorphic to the identity on $\mathcal{K}_2(A)$. □

The pattern that definitions are simpler and theorems more elegant and smooth in the case of partial functions extends to the study of sub-pcas of $\mathcal{B}(A)$. First of all, we have the following proposition.

Proposition 5.3 *The preorder $\leq_T$ on partial functions $A \to A$ (relative to a partial combinatory algebra structure on A) has binary joins.*

Proof Given partial functions f and g define $f \sqcup g$ by

$$(f \sqcup g)(y) \;\simeq\; \textsf{If } p_0 y \textsf{ then } f(p_1 y) \textsf{ else } g(p_1 y).$$

So $(f \sqcup g)([\top, x]) \simeq f(x)$ and $(f \sqcup g)([\bot, x]) \simeq g(x)$. It is left to the reader that $f \sqcup g$ is a join for f, g with respect to $\leq_T$. □

One can now simply define an *ideal* of $\mathcal{B}(A)$ to be a downward closed set which is also closed under $\sqcup$. Given a subset B of $\mathrm{Ptl}(A, A)$, let us write $B_{\subseteq}$ for the set

$$B_{\subseteq} \;=\; \{f \in \mathrm{Ptl}(A, A) \mid \text{there is } g \in B \text{ such that } f \subseteq g\}.$$

Furthermore, let us write $\bar{a}$ for the partial function $x \mapsto ax$, for $a \in A$.

Proposition 5.4 *Let A be a partial combinatory algebra and suppose $\mathcal{B}(A)$ is compatible with A. Suppose $B \subset \mathrm{Ptl}(A, A)$ is nonempty and closed under the application function.*

(i) *If B is downward closed with respect to $\leq_T$, then B is an elementary sub-pca of $\mathcal{B}(A)$.*
(ii) *If for every $a \in A$ we have $\bar{a} \in B$, then $B_{\subseteq}$ is an ideal of $\mathcal{B}(A)$.*

Proof The first item is the remark made before (third paragraph of this section) that in the partial case we can choose k and s to be representable in A. Then, if $\beta \in B$ is arbitrary, we have $k, s \leq_T \beta$; hence, $k, s \in B$ since B is downward closed.

For the second item, first we remark that there is an element $a \in A$ such that for any $\gamma_1, \gamma_2 \in \mathrm{Ptl}(A, A)$, $\bar{a}\gamma_1\gamma_2 = \gamma_1 \sqcup \gamma_2$ in $\mathcal{B}(A)$. This is left to the reader. So from the hypotheses on B it follows that B is closed under $\sqcup$. Since application in $\mathcal{B}(A)$ is monotone in both variables with respect to $\subseteq$, it follows that also $B_{\subseteq}$ is closed under $\sqcup$.

Next we see that $B_{\subseteq}$ is downward closed with respect to $\leq_T$. Suppose $\beta' \in B$, $\beta \subseteq \beta'$, and $\gamma \leq_T \beta$. We need to show $\gamma \in B_{\subseteq}$. Since $\gamma \leq_T \beta$, there is an $a \in A$ satisfying $a \cdot^{\beta} x = \gamma(x)$ for all $x \in \mathrm{dom}(\gamma)$. Then also $a \cdot^{\beta'} x = \gamma(x)$ for all $x \in \mathrm{dom}(\gamma)$. But this means that $\gamma \subseteq \bar{a}\beta'$. Since B is closed under application and $\bar{a}, \beta' \in B$ by assumption, $\gamma \in B_{\subseteq}$ as desired. □

We also have the following generalization of the factorization theorem.

Theorem 5.5 *Let A be a pca and $\mathcal{B}(A)$ compatible with A. Suppose B is an elementary sub-pca of $\mathcal{B}(A)$ containing the (simplified) function ρ of the proof of Proposition 4.1 and all constant functions $\hat{a}$. Let $\gamma : A \to B$ be the decidable applicative morphism $\gamma(a) = \{\hat{a}\}$. Then for any decidable applicative morphism $\delta : A \to C$ such that every partial function in B is representable with respect to δ, there is an applicative morphism $\varepsilon : B \to C$ satisfying $\varepsilon\gamma \cong \delta$.*

I would like to conclude this paper by applying some of its ideas to a specific case. If A is a pca and $\mathcal{B}(A)$ is compatible with A, let

$$T(A) = \{\bar{a} \mid a \in A\}$$

where $\bar{a}$ is as defined just preceding Proposition 5.4. The following proposition generalizes the remark in [11], 1.4.9, that the combinatory algebra $\mathcal{B}$ of [9] has a sub-pca of partial recursive functions, which is in fact an elementary sub-pca because k and s can be taken to be partial recursive.

Proposition 5.6 *$T(A)$ is an elementary sub-pca of $\mathcal{B}(A)$. In particular, $T(A)$ is a total combinatory algebra.*

Proof This is like the proof of Theorem 4.3 (and its analogue Theorem 5.5), where we show that given a pca B and a decidable applicative morphism δ from A to B, there is a uniform way of coding up the "interrogation-type" application $(\alpha, \beta) \mapsto \alpha\beta$ of $\mathcal{B}(A)$ within B, given elements of B which represent α and β with respect to δ.

In other words, there is an element $\varphi \in A$ such that, if we denote the application in $\mathcal{B}(A)$ by $\alpha \bullet \beta$, we have for each $a, b \in A$ that φab is defined and

$$\overline{\varphi ab} = \bar{a} \bullet \bar{b}.$$

This shows that $T(A)$ is closed under the application in $\mathcal{B}(A)$ and since k and s in $\mathcal{B}(A)$ can be chosen to be representable in A, we see that $T(A)$ is an elementary sub-pca of $\mathcal{B}(A)$. □

Let $\gamma : A \to T(A)$ be given by $\gamma(a) = \{\hat{a}\}$. Theorem 5.5 gives us a diagram

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\gamma"'] & & A \\ & T(A) \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

which commutes up to isomorphism: $\varepsilon\gamma \cong \mathrm{id}_A$. Here $\varepsilon(\alpha)$ is the set

$$\{a \in A \mid a \text{ represents } \alpha\} \;=\; \{a \in A \mid \alpha \subseteq \bar{a}\}.$$

However, there is another decidable applicative morphism $\varepsilon' : T(A) \to A$, defined by

$$\varepsilon'(\alpha) \;=\; \{a \in A \mid \alpha = \bar{a}\}.$$

We have the following facts, the first of which can easily be checked by concrete calculation:

(i) $\varepsilon'\gamma \cong \mathrm{id}_A$,
(ii) $\gamma\varepsilon' \preceq \mathrm{id}_{T(A)}$.

For the second fact, pick an element $b \in A$ such that for all $x, v \in A$ one has

$$\begin{array}{rcl} b(\langle x\rangle) & = & \langle \bot, k\rangle \\ b(\langle x, v\rangle) & \simeq & \langle \top, vx\rangle. \end{array}$$

Then one checks that in $T(A)$ one has $\bar{b}\hat{a} = \bar{a}$ for all a, from which one readily deduces the stated fact.

We have therefore an adjunction $\gamma \dashv \varepsilon'$ in the 2-category PCA, and by the theory of geometric morphisms between realizability toposes in chapter 2 of [11] this gives rise to a geometric morphism,

$$\mathrm{RT}(T(A)) \to \mathrm{RT}(A),$$

between the corresponding realizability toposes. Because $\varepsilon'\gamma \cong \mathrm{id}_A$, this geometric morphism is a surjection. We have obtained the following theorem.

Theorem 5.7 *Every realizability topos is a geometric quotient of a realizability topos on a total combinatory algebra.*

6 Topics for Further Research

6.1 Modest sets over $\mathcal{K}_2(A)$ and $\mathcal{B}(A)$ Given a partial combinatory algebra A, the category of *modest sets over* A is defined as follows: objects are pairs $(U, \sim)$ where $U \subseteq A$ and $\sim$ is an equivalence relation on U; maps $(U, \sim) \to (V, \approx)$ are functions $\varphi : U/{\sim} \to V/{\approx}$ such that there is an element a of A satisfying the following: for each $b \in U$, ab is defined and an element of $\varphi([b])$ (where $[b]$ denotes the $\sim$-equivalence class of b).

In the case that $A = \mathcal{K}_2$ with underlying set $\mathbb{N}^{\mathbb{N}}$, every modest set over A can be regarded as a topological space (the quotient topology on $U/{\sim}$, where U is topologized as subspace of Baire space $\mathbb{N}^{\mathbb{N}}$), and, moreover, every map of modest sets is continuous with respect to these topologies. Since every category of modest sets over a partial combinatory algebra is cartesian closed, this gives rise to cartesian closed subcategories of the category of topological spaces. This is exploited in [2] and [1].

It is not so clear how $\mathcal{K}_2(A)$ relates to this for uncountable A. Certainly we can topologize modest sets over $\mathcal{K}_2(A)$ in the obvious way sketched above, and we obtain cartesian closed full subcategories of the category of modest sets over $\mathcal{K}_2(A)$ that are (nonfull) subcategories of the category of topological spaces.

The question is whether there is, from the topological point of view, anything of interest to say about such subcategories. Is there any way to characterize the sequential maps topologically? At present I am inclined to think that there is not a topology on A^A and a topologically definable class of domains D (comparable to the

G_δ-sets for $\mathcal{K}_2$) such that the sequential maps are precisely the partial continuous ones with domain in D. Note, for example, that partial sequential functions don't enjoy the "pasting property."

6.2 Relation to Graph Models It is well known (first noted in [2]; see also [11], Examples 4 and 5 of Section 1.5) that there are applicative morphisms $\gamma : \mathcal{P}(\omega) \to \mathcal{K}_2$ and $\iota : \mathcal{K}_2 \to \mathcal{P}(\omega)$ such that $\gamma\iota \simeq \mathrm{id}_{\mathcal{K}_2}$ and $\iota\gamma \preceq \mathrm{id}_{\mathcal{P}(\omega}$, giving rise to a surjective geometric morphism from the realizability topos on $\mathcal{P}(\omega)$ to the one on $\mathcal{K}_2$. It is natural to wonder whether this extends to the case of $\mathcal{P}(A)$ and $\mathcal{K}_2(A)$ for uncountable A. As far as I can see, the construction of γ uses the countability of ω in an essential way, however.

Notes

1. In [9], I called these "dialogues." Now I find they are far too one-sided for that name.

2. There is a similar problem for defining s in $\mathcal{K}_2$, and the proof in [11], Lemma 1.4.1 is inaccurate.

3. This notation was suggested to me by the first referee.

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