What is Synthetic Domain Theory?

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Synopsis

1. "Synthetic"

Antonym: "Analytic", as in:

Kant: synthetic a priori truths, vs. analytic truths

synthetic a priori truths — true by 'insight':
Kant: Euclidean Geometry
Poincaré: mathematical induction

analytic truths — true by virtue of the meaning of words, or by Logic

19th century: synthetic geometry vs. analytic geometry

synthetic geometry — axioms for constructions

analytic geometry — algebra on coordinates

Also: "synthetic reasoning" in differential geometry
Characteristics: free use of 'infinitely small' (or large) numbers, reasoning about manifolds without mention of atlases or charts.

<u>Example</u> (Euler) Proof of $\sin z = z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2 \pi^2}\right)$

Start of proof: 'for infinitely large n',

$$2 \cdot \sinh(x) = \left(1 + \frac{x}{n}\right)^n - \left(1 - \frac{x}{n}\right)^n$$

Then use: $a^n - b^n = (a-b)(a - \varepsilon_1 b) \cdot \ldots \cdot (a - \varepsilon_{n-1} b)$
where $1, \varepsilon_1, \ldots, \varepsilon_{n-1}$ are (complex) solutions to $z^n = 1$

What is meant nowadays by 'synthetic'?

Loosely: "reason with axioms"

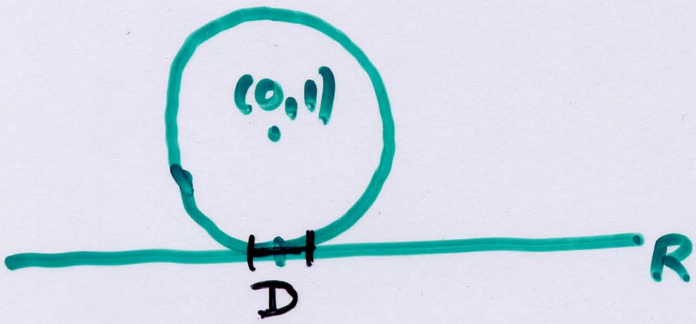
An example: a glimpse of Synthetic Analysis (or: Synthetic Differential Geometry)

Starting point: the real line R as 'given', with points $0, 1$

On R we have (by geometric constructions):

$+, \cdot, -, \div n \ (n \in \mathbb{N})$ R is a $\mathbb{Q}$-algebra

Define: $D = \{x \in R : x^2 = 0\}$



<u>Axiom</u>: for every function $f: D \to R$ there is a <u>unique</u> $b \in R$ such that for all $d \in D$:
$$f(d) = f(0) + b \cdot d$$

<u>Consequence 1</u>: $D \neq \{0\}$ (uniqueness of b)

<u>Consequence 2</u>: Classical Logic fails

Suppose now $f: R \to R$ is a function.

Then for each x, consider $d \mapsto f(x+d) : D \to R$

By the axiom, there is a unique $b = f'(x)$ such that for all $d \in D$: $f(x+d) = f(x) + d \cdot b$

It is easy to derive that:

$(f+g)'(x) = f'(x) + g'(x)$ $(f \cdot g)'(x) = f(x) g'(x) + f'(x) g(x)$

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ (Chain rule)

And: Taylor series, partial derivatives, smooth manifolds, integration, flows, ...

2. A review of some elements of Domain Theory

a. ω-cpos

<u>Definition</u>. ω-cpo: poset $(X, \leq)$ such that for every ω-chain $x_0 \leq x_1 \leq x_2 \leq \dots$ there is a least upper bound $\bigsqcup_{n\in\omega} x_n$:

i) for all i, $x_i \leq \bigsqcup_{n\in\omega} x_i$

ii) if for all i, $x_i \leq y$, then $\bigsqcup_{n\in\omega} x_i \leq y$

If $(X, \leq)$ and $(Y, \leq)$ are ω-cpos then $f : X \to Y$ is <u>continuous</u> if for every ω-chain $x_0 \leq x_1 \leq \dots$ in X, $f(x_0) \leq f(x_1) \leq \dots$ is an ω-chain in Y and

$$f\Big(\bigsqcup_n x_n\Big) = \bigsqcup_n f(x_n)$$

CPO is the category of ω-cpos and continuous maps

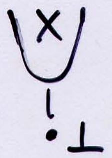
The category CPO is <u>complete</u>:
if $\{X_i : i \in I\}$ is a family of ω-cpos then

$$\prod_{i\in I} X_i \text{ is an } \omega\text{-cpo}$$

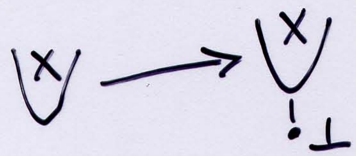
The category CPO is <u>cartesian closed</u>:
for two ω-cpos X and Y, let X^Y be the set of continuous functions $Y \to X$. Set $f \leq g$ if for all $y \in Y$, $f(y) \leq g(y)$. Then X^Y is an ω-cpo

There is a <u>monad</u> $L : \mathrm{CPO} \to \mathrm{CPO}$ ("Lifting")

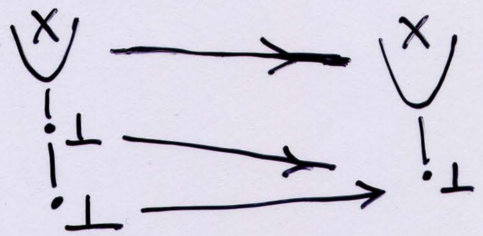
LX is X with a new least element added:



For each X, we have:



$$X \xrightarrow{\eta_X} L(X)$$



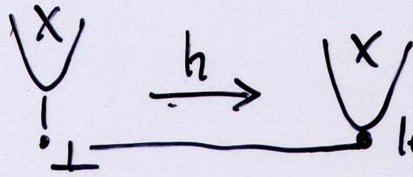
$$L(L(X)) \xrightarrow{\mu_X} L(X)$$

An <u>algebra</u> for L is a cpo X together with a continuous map $L(X) \xrightarrow{h} X$ such that:

i) $h(\eta_X(x)) = x$

ii)
$$\begin{tikzcd} L^2X \arrow[r, "L(h)"] \arrow[d, "\mu_X"'] & LX \arrow[d, "h"] \\ LX \arrow[r, "h"'] & X \end{tikzcd}$$
commutes

Fact: X is an L-algebra if and only if X has a <u>least element</u>:



least element

Fact: If X is an ω-cpo with least element $\perp$, then every continuous $f : X \to X$ has a <u>least fixed point</u>, namely $\bigsqcup_{n \in \omega} f^n(\perp)$ $\quad (\perp \le f(\perp) \le f^2(\perp) \le \dots)$

Fact: if X and Y are ω-cpo's with $\bot$, then Y^X is an ω-cpo with $\bot$

Fact: the function $fix : X^X \to X$ such that $fix(f) =$ the least fixed point of f, is continuous.

Embedding - Projection pairs

A pair of ~~[redacted]~~ continuous functions $X \underset{i}{\overset{r}{\rightleftarrows}} Y$ is an <u>embedding - projection pair</u> if

$$ri(x) = x \quad , \quad x \in X$$
$$ir(y) \leq y \quad , \quad y \in Y$$

Then: if X and Y have $\bot$, both i and r preserve $\bot$

[redacted]

<u>Theorem</u> (limit - colimit coincidence)

Suppose we have a chain of embedding - projection pairs

$$\begin{tikzcd} P_1 \arrow[r, shift right, "i_1"'] & P_2 \arrow[l, shift right, "r_1"'] \arrow[r, shift right, "i_2"'] & P_3 \arrow[l, shift right, "r_2"'] \arrow[r, shift right] & \cdots \arrow[l, shift right] \end{tikzcd}$$

and all P_i are ω-cpo's with $\bot$.

Then there is an ω-cpo with $\bot$, P, and embedding - projection pairs $P_n \underset{j_n}{\overset{q_n}{\rightleftarrows}} P$ such that:

$$\begin{tikzcd} P_1 \arrow[r, "i_1"] \arrow[dr, "j_1"'] & P_2 \arrow[r, "i_2"] \arrow[d, "j_2"] & \cdots \arrow[dl] \\ & P & \end{tikzcd}$$

is a <u>colimit</u> in CPO

and:

$$\begin{tikzcd} & P \arrow[dl, "q_1"'] \arrow[d, "q_2"] \arrow[dr] & \\ P_1 & P_2 \arrow[l, "r_1"] & \cdots \arrow[l, "r_2"] \end{tikzcd}$$

is a <u>limit</u> in CPO

And in the category $CPO_\perp^{EP}$ of ω-cpo's with $\perp$ and embedding-projection pairs as maps,

$$\begin{tikzcd} P_1 \arrow[r] \arrow[dr] & P_2 \arrow[r] \arrow[d] & \cdots \\ & P & \cdots \end{tikzcd}$$

is a colimit

<u>Theorem</u> Suppose $F : (CPO_\perp^{op})^m \times (CPO_\perp)^n \to CPO_\perp$ is 'locally continuous', that is: its action on maps preserves l.u.b.'s of ω-chains.

Then there is a functor $\tilde{F} : (CPO_\perp^{EP})^{m+n} \to CPO_\perp^{EP}$ satisfying:

- the action of $\tilde{F}$ on objects is the same as F
- $\tilde{F}$ preserves colimits of ω-chains

<u>Application</u>

Consider $F : CPO_\perp^{op} \times CPO_\perp \to CPO_\perp$

$$F(Y, X) = X^Y$$

Have the composite

$$G : CPO_\perp^{EP} \xrightarrow{\Delta} CPO_\perp^{EP} \times CPO_\perp^{EP} \xrightarrow{\tilde{F}} CPO_\perp^{EP}$$

$$X \longmapsto X^X$$

Starting with any ω-cpo with $\perp$, P_0, one can construct an embedding-projection pair

$$P_0 \rightleftarrows P_1 = P_0^{P_0}$$

This gives a chain

$$P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \longrightarrow \cdots \quad \text{in } CPO_\perp^{EP}$$

with $P_{i+1} = G(P_i)$ and $f_{i+1} = G(f_i)$

If P is the colimit of this chain, then since G preserves colimits of chains, $P \cong G(P) = P^P$

$P \cong P^P$ gives a nontrivial model of the untyped λ-calculus

In this way one can "solve recursive domain equations" (Meaning, find ω-cpo's X satisfying $X \cong \Phi(X)$ where Φ represents an appropriate functor)

Also, solve systems like

$$\begin{aligned} X_1 &\cong \Phi_1(X_1, \dots, X_n) \\ X_2 &\cong \Phi_2(X_1, \dots, X_n) \\ &\vdots \\ X_n &\cong \Phi_n(X_1, \dots, X_n) \end{aligned}$$

3. Why partial orders?

Consider the language PCF

<u>Types</u>: B (Booleans) | N (integers) | Type $\to$ Type

variables $x^\sigma, \ldots$

<u>Terms</u> include

$t, f : B$ (true, false)

If ... then ... else ..: $B \to (N \to (N \to N))$ (definition by cases)

$Y_\sigma : (\sigma \to \sigma) \to \sigma$ (fix point)

(and much more)

+ λ-terms $\lambda x^\sigma M : \sigma \to \tau$

<u>Operational semantics</u>: given by a reduction relation $\mapsto$, including

$$\text{If } t \text{ then } M \text{ else } N \;\mapsto\; M$$

$$Y_\sigma M \;\mapsto\; M(Y_\sigma M)$$

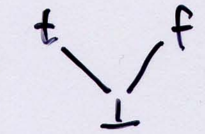
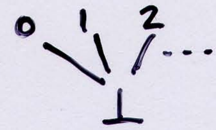
$$(\lambda x^\sigma M) N \;\mapsto\; M[N/x]$$

etc.

Operational preorder: say $M \precsim N$ if, whenever $C[M] \mapsto c$ and c is a constant of type B or N, then $C[N] \mapsto c$

<u>Denotational semantics</u>: interpretation of terms in ω-cpo's with $\perp$

For each type σ have an ω-cpo with $\perp$, $[\![\sigma]\!]$:

$[\![B]\!] =$  $[\![N]\!] =$  $[\![\sigma \to \tau]\!] = [\![\tau]\!]^{[\![\sigma]\!]}$

Every closed term M of type σ is interpreted by an element of $[\![\sigma]\!]$, e.g. for M of type $\sigma \to \sigma$,

$$[\![Y_\sigma M]\!] = \mathrm{fix}([\![M]\!])$$

Denotational preorder: $M \sqsubseteq N$ if $[\![M]\!] \leq [\![N]\!]$

We say:

$[\![\cdot]\!]$ is <u>adequate</u> if $M \sqsubseteq N$ implies $M \underset{\sim}{\sqsubset} N$

$[\![\cdot]\!]$ is <u>fully abstract</u> if $M \underset{\sim}{\sqsubset} N$ implies $M \sqsubseteq N$

4. Synthetic Domain Theory

Goal: try to find a category of just sets, with all its functions, which has properties suitable to carry out the constructions we did for ω-cpo's.

Basic assumption

(i) For every set X, a set $\mathcal{O}(X)$ of "open subsets" of X is given, such that if $f: X \to Y$ is a function and $U \in \mathcal{O}(Y)$, then $f^{-1}(U) = \{x \in X : f(x) \in U\} \in \mathcal{O}(X)$

(ii) There is a set Σ and an open subset $\top$ of Σ such that for every X and every $U \in \mathcal{O}(X)$, there is a <u>unique</u> $c_U : X \to \Sigma$ such that $U = c_U^{-1}(\top)$

(ii) implies: Σ can be identified with a subset of $\mathcal{P}(\{*\})$, that is, with a set of <u>propositions</u>

$$\left(\begin{array}{rcl} \mathcal{P}(\{*\}) & \cong & \text{Propositions via} \\ \alpha & \longmapsto & [* \in \alpha] \\ \{* : p\} & \longleftarrow\!\shortmid & p \end{array} \right)$$

We write $U \subset_\circ X$ for $U \in \mathcal{O}(X)$

Axiom 1 i) For every set X, $X \subset_\circ X$

ii) For every set X, $\emptyset \subset_\circ X$

iii) For every set X, if $Y \subset_\circ X$ and $U \subset_\circ Y$, then $U \subset_\circ X$

i), $\forall X\ (X \subset_0 X)$ implies, that under $\Sigma \subset \mathcal{P}(\{*\})$, $\top \in_0 \Sigma$ can be identified with $\{\mathrm{true}\}$

ii), $\forall X\ (\emptyset \subset_0 X)$ is equivalent to: $\overset{\text{false}}{\bot} \in \Sigma$

i) + iii), $\forall X Y U\ (U \subset_0 Y \subset_0 X \Rightarrow U \subset_0 X)$ is equivalent to the <u>dominance axioms</u> (Rosolini):

a) $\mathrm{true} \in \Sigma$

b) $\big(p \in \Sigma \wedge p \Rightarrow (q \in \Sigma)\big) \Rightarrow ((p \wedge q) \in \Sigma)$

We shall be interested in <u>partiality</u>: partial functions $X \rightharpoonup Y$ with <u>open</u> domain

First, a remark on dominances

Suppose A is a set. We write $\tilde{A}$ for the set

$$\{B \subset A : B \text{ has at most one element}\}$$

For $\alpha \in \tilde{A}$ we write $I(\alpha)$ for "α has an element", and $\downarrow\alpha$ for the unique element of α, if $I(\alpha)$.

Suppose $\Phi \subset A$ is a subset such that:

1) $\exists a \, (a \in \Phi)$

2) For all $a \in A$ and all $\Psi \subset A$, if $(a \in \Phi \Rightarrow \exists b \, (b \in \Psi))$ then there is an $\alpha \in \tilde{A}$ such that

$$(a \in \Phi \Rightarrow I(\alpha) \wedge \sim \downarrow\alpha \in \Psi)$$

3) For all $a \in A$ and all $\alpha \in \tilde{A}$, if $(a \in \Phi \Rightarrow I(\alpha))$ then there is $b \in A$ such that $((a \in \Phi \wedge \downarrow\alpha \in \Phi) \Leftrightarrow b \in \Phi)$

Then the set $\Sigma = \{p : \exists a \in A \, (p \leftrightarrow a \in \Phi)\}$ is a dominance:

by 1) $\text{true} \in \Sigma$

Suppose $p \in \Sigma$ and $p \Rightarrow (q \in \Sigma)$. Let $a \in A$ such that $p \leftrightarrow a \in \Phi$; then

$$a \in \Phi \Rightarrow \exists b \, (q \leftrightarrow b \in \Phi)$$

By 2), there is $\alpha \in \tilde{A}$ such that

$$a \in \Phi \Rightarrow I(\alpha) \wedge (q \leftrightarrow \downarrow\alpha \in \Phi)$$

By 3), there is $b \in A$ such that

$$((a \in \Phi \wedge \downarrow\alpha \in \Phi) \Leftrightarrow b \in \Phi);$$

then $p \wedge q \Leftrightarrow b \in \Phi$

Example: A a set of computations of some type, and $\Phi \subset A$ the set of terminating computations

Return to partial maps with open domain:

$$\begin{array}{ccc} U & \subseteq_o & X \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

Definition. $L(Y) = \{\alpha \in \tilde{Y} : I(\alpha) \in \Sigma\}$

There are maps $\eta_Y : Y \longrightarrow L(Y)$ $\quad (\eta_Y(y) = \{y\})$

and $\mu_Y : L^2(Y) \longrightarrow L(Y)$ $\quad (\mu_Y(A) = \bigcup A)$

For $f : X \longrightarrow Y$ define $L(f) : L(X) \longrightarrow L(Y)$ by

$$L(f)(\alpha) = \{f(x) : x \in \alpha\}$$

Observe: For every partial map as above there is a unique total function $\tilde{f} : X \longrightarrow L(Y)$ such that $(\tilde{f})^{-1}(Y) = U$. Namely,

$$\tilde{f}(x) = \{fx : x \in U\}$$

L is a monad, similar to the Lifting monad for cpo's.

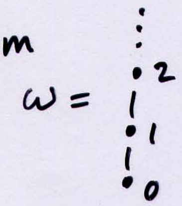
Classically, $\Sigma = \begin{array}{c} \bullet\,\text{true} \\ | \\ \bullet\,\text{false} \end{array}$

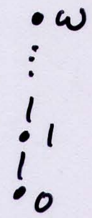
Classically, $LX = \begin{array}{c} X \\ | \\ \bullet\,\emptyset \end{array}$

Definition. An <u>object with $\perp$</u> is an algebra for the monad L.

A <u>strict map</u> between objects with $\perp$ is an L-algebra homomorphism

Complete objects

In Posets, an ω-chain is an order-preserving map from  into an object.

Consider also $\omega+1$: 

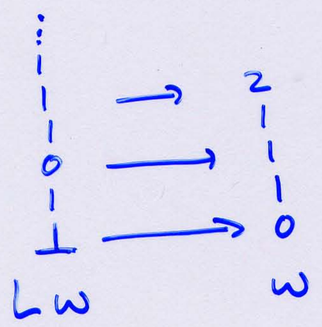
X is a cpo if and only if every monotone map $f : \omega \to X$ has a unique continuous extension to $\bar{f} : \omega+1 \longrightarrow X$

Definition. A weak L-algebra is an object X together with a map $LX \xrightarrow{h} X$

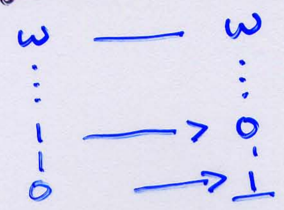
A morphism of weak L-algebras $(X, h) \to (Y, k)$ is a function $f : X \longrightarrow Y$ such that

$$\begin{tikzcd} LX \arrow[r, "Lh"] \arrow[d, "h"'] & LY \arrow[d, "k"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes

In Posets,  is the initial weak L-algebra

A weak L-coalgebra is $(X \xrightarrow{h} LX)$.

In Posets,  is the final weak L-coalgebra

Lambek's Lemma If $\mathcal{C}$ is a category and $F : \mathcal{C} \to \mathcal{C}$ a functor, then if $(X, FX \xrightarrow{h} X)$ is an initial weak F-algebra, h is an isomorphism

Proof. Given $(X, FX \xrightarrow{h} X)$ consider the weak F-algebra $(FX, F^2X \xrightarrow{Fh} FX)$. By initiality of $(X, FX \xrightarrow{h} X)$ there is a unique $i : X \to FX$ such that

$$\begin{tikzcd} FX \arrow[r, "Fi"] \arrow[d, "h"'] & F^2X \arrow[d, "Fh"] \\ X \arrow[r, "i"] & FX \end{tikzcd}$$

commutes

Clearly, also

$$\begin{tikzcd} F^2X \arrow[r, "Fh"] \arrow[d, "Fh"'] & FX \arrow[d, "h"] \\ FX \arrow[r, "h"'] & X \end{tikzcd}$$

commutes

Composing, $hi : X \to X$ is a morphism of weak F-algebras from (X, h) to itself. Since (X, h) is initial, $hi = id_X$.

From the first diagram, we see that

$$ih = Fh \circ Fi = F(hi) = F(id_X) = id_{FX}$$

So h is an isomorphism with inverse i.

$\square$

Of course, the same holds for a final weak coalgebra

In our general case of sets and functions, and

$LX = \{\alpha \in \tilde{X} : I(\alpha) \in \Sigma\}$

do initial weak L-algebra and final weak L-coalgebra exist?

Yes. Let $N = \{0, 1, \ldots\}$ the set of natural numbers.

Let $F = \{\psi \in \Sigma^N : \forall n \in N \, (\psi(n+1) \Rightarrow \psi(n))\}$

Define $\tau : F \longrightarrow LF$ by

$$\tau(\psi) = \{\lambda n. \psi(n+1) : \psi(0)\}$$

Then (F, τ) is a final weak L-coalgebra:

If $(X, \sigma : X \to LX)$ is a weak L-coalgebra there is a unique homomorphism of weak L-coalgebras $f : X \longrightarrow F$ given by

$$f(x)(0) = I(\sigma(x))$$
$$f(x)(n+1) = I(\sigma(x)) \wedge f(\downarrow \sigma(x))(n)$$

L also has an initial weak algebra

$$LI \xrightarrow{\sigma} I$$

(Jibladze)

$I = \{\psi \in F : \forall \text{ propositions } \phi$
$\qquad \forall n \in N \, ((\psi(n) \to \phi) \to \phi) \longrightarrow \phi\}$

$\tau : F \longrightarrow LF$ is an isomorphism, and $\sigma : LI \to I$ is the restriction of τ^{-1} to LI

- For I, we have an induction principle:

  if $A \subset I$ and

  $\forall \psi \in I\ [\ \exists n \in N\ (\psi(n) \to \psi \in A) \to \psi \in A\ ]$

  then $A = I$

- The following holds for $\psi \in F$:

  $\exists n \in N\ \neg\psi(n) \Rightarrow \psi \in I \Rightarrow \neg\neg\exists n\ \neg\psi(n)$

  Classical pictures:

I	F
	$(\top,\top,\top,\dots)$
$\vdots$	$\vdots$
$(\top,\top,\bot,\bot,\dots)$	
$(\top,\bot,\bot,\dots)$	$(\top,\bot,\bot,\dots)$
$(\bot,\bot,\bot,\dots)$	$(\bot,\bot,\dots)$

By the initial-weak algebra property we have $\iota : I \to F$ (in fact, the embedding) such that

$$\begin{tikzcd} LI \arrow[r, "L\iota"] \arrow[d, "\sigma"'] & LF \arrow[d, "\sigma"] \\ I \arrow[r, "\iota"] & F \end{tikzcd}$$

commutes.

Definition. X is <u>complete</u> if the function

$$X^{\iota} : X^F \to X^I \text{ is an isomorphism}$$

That is, if every function $I \to X$ extends uniquely to a function $F \to X$

Example $\{*\}$ is complete (obvious)
I is not complete, since $I \neq F$

Theorem. Suppose X is an object with $\perp$ (an L-algebra). If X is complete, then every function $g: X \to X$ has a fixed point.

Proof. There are maps

$$\rho: LI \longrightarrow I \qquad \rho(\alpha)(n) = I(\alpha) \wedge \downarrow\alpha(n)$$

$$s: I \longrightarrow I \qquad s(\varphi)(n) = \begin{cases} \top & \text{if } n=0 \\ \varphi(n-1) & \text{if } n>0 \end{cases}$$

One can prove: (I, ρ) is an L-algebra. Moreover, for every L-algebra $(LX \xrightarrow{a} X)$ and every $g: X \to X$, there is a unique function $h: I \to X$ such that both

$$\begin{tikzcd} LI \arrow[r, "Lh"] \arrow[d, "\rho"'] & LX \arrow[d, "a"] \\ I \arrow[r, "h"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I \arrow[r, "h"] \arrow[d, "s"'] & X \arrow[d, "g"] \\ I \arrow[r, "h"] & X \end{tikzcd} \quad \text{commute}$$

Since X is complete, h extends uniquely to $\bar{h}: F \longrightarrow X$.
Also, $s: I \to I$ extends to $\bar{s}: F \to F$ (same definition)
By uniqueness of $\bar{h}$,

$$\begin{tikzcd} F \arrow[r, "\bar{h}"] \arrow[d, "\bar{s}"'] & X \arrow[d, "g"] \\ F \arrow[r, "\bar{h}"] & X \end{tikzcd} \quad \text{commutes}$$

But in F, $\bar{s}$ has a fixed point $\omega = (\top, \top, \cdots)$
Hence, $g(\bar{h}(\omega)) = \bar{h}(\bar{s}(\omega)) = \bar{h}(\omega)$, so $\bar{h}(\omega)$ is a fixed point for g. $\square$

Easy to see:

- Any limit of a diagram of complete objects is complete, in particular: if X is complete, so is X^Y for each Y. Moreover, retracts of complete objects are complete

Can we find a category of complete objects which has similar properties as CPO?

No.

Because, using classical logic, if X is complete hence every function $X \to X$ has a fixed point, $X \cong \{*\}$.

The theory up to now, although consistent with classical logic, seems rather empty...

<u>Axiom 2</u> Σ is complete

Axiom 2 is classically inconsistent, but consistent with intuitionistic logic!

Axiom 2 is equivalent to: F is complete.

For, Σ is a retract of F

and F is a retract of Σ^N: by $\Phi : \Sigma^N \to F$

$$\Phi(\varphi)(n) = \bigwedge_{i=0}^{n} \varphi(n)$$

Now for a category of (pre) domains

Definition (Freyd) A category $\mathcal{C}$ is called algebraically compact if for every endofunctor $T: \mathcal{C} \to \mathcal{C}$ there exist an initial weak T-algebra $TI \xrightarrow{\sigma} I$ and a final weak T-coalgebra $F \xrightarrow{\tau} LF$, and moreover, the canonical map

$$\begin{tikzcd} TI \arrow[r, "T\imath"] \arrow[d, "\sigma"'] & TF \arrow[d, "\tau^{-1}"] \\ I \arrow[r, "\imath"] & F \end{tikzcd}$$

is an isomorphism

Big advantage of the definition: it is self-dual (if $\mathcal{C}$ is algebraically compact, so is $\mathcal{C}^{op}$)

Theorem (Freyd) If the categories $\mathcal{C}$ and $\mathcal{D}$ are algebraically compact, so is $\mathcal{C} \times \mathcal{D}$.

Hence: if $\mathcal{A}$ is alg. compact, so is $(\mathcal{A}^{op})^n \times \mathcal{A}^m$.

Therefore recursive domain equations can be solved in $\mathcal{A}$, as shown by Fiore (thesis).

We are interested in complete, algebraically compact categories. Classically, this is hard to satisfy:

1. Easy: every algebraically compact category has a zero object (an object which is both initial and final)
2. Every algebraically compact preorder is trivial
3. (Classically) Every complete, algebraically compact category is a preorder

But: in models of intuitionistic set theory, nontrivial complete, algebraically compact categories exist!

Categories of predomains These should satisfy the properties of CPO, and preferably more:

- full subcategories of Set
- the objects are complete
- the category is complete
- the category is closed under L

Nice to have, moreover:

- the category is (essentially) small (for interpreting polymorphic type theories)
- the category is algebraically compact

Domains, then, are predomains with $\perp$

Examples.

1) Replete objects (Hyland, Taylor, Phoa)

Call a monomorphism $m: A \rightarrowtail B$ Σ-dense if $\Sigma^m : \Sigma^B \to \Sigma^A$ is an isomorphism

X is replete if for every Σ-dense mono m, $X^B \xrightarrow{X^m} X^A$ is an isomorphism.

Easy to see:
- full subcategory on replete objects is complete
- every replete object is complete (since $\iota : I \to F$ is Σ-dense)

Moreover, if X is replete so is $L(X)$

And: in the realizability model of set theory, the replete objects form (essentially) a <u>set</u>

2) Regular Σ-posets

For this example, assume a further condition on Σ, namely:

(*) $\forall p \in \Sigma\ (\neg\neg p \Rightarrow p)$

We have, always, a preorder on every set X: $x \leq y$ if for every open $U \subseteq_o X$, if $x \in U$ then $y \in U$. In other words, iff

$$\forall f \in \Sigma^X\ (f(x) \Rightarrow f(y))$$

Let $\phi_X : X \to \Sigma^{\Sigma^X}$ $\quad \phi(x)(f) = f(x)$

Then: $\leq$ is a partial order iff ϕ_X is monic

We say: X is a **Σ-poset** if ϕ_X is monic.

We say: X is a **regular Σ-poset** if ϕ_X is monic and

$$\forall A \in \Sigma^{\Sigma^X} \left(\neg\neg \exists x \in X\, (A = \phi_X(x)) \Rightarrow \exists x \in X\, (A = \phi_X(x)) \right)$$

Theorem [Reus; vO-Simpson] If X is a complete regular Σ-poset, so is $L(X)$

In the realizability model of Set Theory, the regular Σ-posets have been called 'Extensional PER's' or ExPer's

In this model, we have:

Theorem [Freyd-Mulry-Rosolini-Scott]

a) ExPer's form a complete, small category

b) The category $ExPer_0$ on complete ExPers with $\perp$, and $\perp$-pres. maps is algebraically compact

3) Well-complete objects [Longley; Simpson]

This seems to be the most successful notion.

Call X **well-complete** if $L(X)$ is complete.

Facts: i) if X is well-complete, X is complete

ii) if X is well-complete, so is $L(X)$

iii) well-complete objects form a complete category

Simpson (2002; 2004) proves a strong theorem about categories of well-complete objects.

For a correct formulation of it, we need to be a little less informal than up to now about what we mean by "sets"

Sets and Classes

Assume we're in a world of 'classes' (and functions between them) in which there is a universe of sets which is a model of intuitionistic set theory.

In particular we shall need the Axiom of Replacement which says:

If $\mathcal{X}$ is a class, X a set, and $F : X \to \mathcal{X}$ a function, then there is a set Y which contains $\{F(x) : x \in X\}$

Sets are 'small'. We assume we have an apparatus to reason about small things; for example for each class $\mathcal{X}$ we can form

$$\mathcal{P}_s(\mathcal{X}) = \{\mathcal{Y} \subseteq \mathcal{X} \mid \mathcal{Y} \text{ is small}\}$$

Some axioms:

1) $\{*\}$ is small
2) If $\mathcal{X}$ is small, so is $\mathcal{P}_s(\mathcal{X})$
3) If $\mathcal{X}$ is small and $\mathcal{Y} \subseteq \mathcal{X}$ then $\mathcal{Y}$ is small
4) If $\mathcal{X}$ and $\mathcal{Y}$ are small, so is $\mathcal{Y}^{\mathcal{X}}$
5) N is small

Immediate consequences:

- $\mathcal{P}(\{*\}) = \mathcal{P}_s(\{*\})$ is small, hence so is $\Sigma \subset \mathcal{P}(\{*\})$
- $F \subset \Sigma^N$ is small, hence so is $I \subset F$
- If X is small, so is $L(X) \subset \mathcal{P}(\{*\})^X$

Let K be the category of small well-complete objects and partial maps with open domain (equivalently, K is the Kleisli category for L on small well-complete objects)

K generalizes: ω-cpo's with $\perp$ and $\perp$-preserving maps

Theorem (Simpson) K is algebraically compact

"Proof" (a few basic ideas) This is a glorification (higher dimensional generalization) of the proof that every endofunction on a complete object with $\perp$ has a fixed point.

Recall: If X is an L-algebra, $g: X \to X$ then there is a unique L-algebra homomorphism $h: (I, \rho) \to X$ such that

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, "s"'] & X \arrow[d, "g"] \\ I \arrow[r, "h"'] & X \end{tikzcd}$$

commutes.

Essentially, this result is lifted to the level of categories. K_0, the class of objects of K, is an L-algebra, as well as each hom-set $K(A, B)$.

I is a partial order, hence a category, and also an L-algebra in the above sense; $s: I \to I$ is an endofunctor.

There is a notion of 'L-algebra homomorphism' for functors between such 'L-algebra categories'

Then, for each endofunctor $F: K \to K$ there is a unique L-algebra homomorphism functor $H: I \to K$ such that

$$\begin{tikzcd} I \arrow[r, "H"] \arrow[d, "s"'] & K \arrow[d, "F"] \\ I \arrow[r, "H"'] & K \end{tikzcd}$$

commutes

[the construction of H makes essential use of the Replacement axiom]

A notion of 'bilimit' for such H (limit-colimit) is defined. By completeness of K, H has such a bilimit; this will be the carrier of an initial weak algebra – final weak coalgebra for the functor F.

[Note H generalizes (and modifies) the diagram of 'iterates' of F] ■

Application (Simpson) This result has been applied to give a proof of computational adequacy for the language FPC with recursively defined types.

Rosolini and Simpson (2004) go one step further: assuming a small category of domains which is also [redacted] complete, define relationally parametric model for a polymorphic language, and prove computational adequacy.

Concluding this sketch of the theoretical development of synthetic domain theory, the following quote from Rosolini-Simpson 2004:

'This paper reveals synthetic domain theory to be a serious competitor to state-of-the-art techniques in operational semantics'

Models

1. A model in the realizability topos Eff

Our predomains will be $\neg\neg$-separated objects (objects X for which $\forall xy \in X\ (\neg\neg(x=y) \to x=y)$ is true). These can be described as assemblies:

An assembly is a (classical) set X together with a function $|\cdot|_X : X \to \mathcal{P}\mathbb{N}$ taking values in the nonempty subsets of $\mathbb{N}$

A morphism of assemblies $(X, |\cdot|_X) \to (Y, |\cdot|_Y)$ is a function $f: X \to Y$, such that there exist a partial recursive function φ with the property that for all $x \in X$ and all $n \in |x|_X$, $\varphi(n) \in |f(x)|_Y$

The dominance Σ in Eff. Let $K = \{e \mid e \cdot e \downarrow\}$.

$$\text{Now } \Sigma = \{p \mid \exists e \in \mathbb{N}\ (p \leftrightarrow e \in K)\}$$

In Eff, Markov's Principle $(\neg\neg(e \in K) \to e \in K)$ holds; so $\forall p \in \Sigma\ (\neg\neg p \to p)$. In particular, Σ is $\neg\neg$-separated

As assembly, Σ can be represented as

$$\Sigma = (\{\top, \bot\}, |\cdot|_\Sigma) \qquad |\top|_\Sigma = K, \quad |\bot|_\Sigma = \overline{K} = \mathbb{N} - K$$

Given assembly $(X, |\cdot|_X)$ and $X' \subseteq X$, the sub-assembly $(X', |\cdot|_X)$ is an open subset iff there is a recursively enumerable subset A of $\mathbb{N}$

such that:

i) for all $x \in X'$, $|x|_X \subset A$

ii) for all $x \in X - X'$, $|x|_X \cap A = \emptyset$

(Every open subset is of this form)

The lift functor L on assemblies:

$$L(X, |\cdot|_X) = (X \sqcup \{\bot\}, |\cdot|_{LX})$$

$$|x|_{LX} = \{n \in \mathbb{N} : n \cdot n \in |x|_X\}$$

$$|\bot|_{LX} = \overline{K}$$

The final weak L-coalgebra F can be rendered as: $F = (\omega + 1, |\cdot|_F)$

$$|n|_F = \{e \mid W_e = \{m \mid m < n\}\}$$

$$|\omega|_F = \{e \mid W_e = \mathbb{N}\}$$

As we know, $F = \{\varphi \in \Sigma^N \mid \forall n (\varphi(n+1) \Rightarrow \varphi(n))\}$

From Jibladze's formula for I it follows that

$$\{\varphi \in F \mid \exists n \neg\varphi(n)\} \subseteq I \subseteq \{\varphi \in F \mid \neg\neg\exists n \neg\varphi(n)\}$$

In fact, for Eff one can show that

$$I = \{\varphi \in F \mid \neg\neg\exists n \neg\varphi(n)\}$$

which means that I is a sub-assembly of F:

$$I = (\omega, |\cdot|_F)$$

Let us show that Σ is complete

Suppose $\phi : I \to \Sigma$. Let $X = \phi^{-1}(\{\top\})$

X corresponds to an open subassembly of I, so there is a recursively enumerable set A such that:

(1) $n \in X \Rightarrow \{e \mid W_e = [0, \ldots, n-1]\} \subset A$

(2) $n \notin X \Rightarrow \{e \mid W_e = [0, \ldots, n-1]\} \cap A = \emptyset$

<u>Lemma</u> If $n \in X$, $n+1 \in X$

<u>Proof</u> Suppose $A = W_a$. By recursion theorem, pick e such that

$$e \cdot k \simeq \begin{cases} 0 & \text{if } k < n \\ a \cdot e & \text{if } k = n \\ \text{undefined} & \text{else} \end{cases}$$

If $a \cdot e \uparrow$, $W_e = [0, \ldots, n-1]$ so $n \notin X$; hence if $n \in X$, $a \cdot e \downarrow$ and $W_e = [0, \ldots, n]$, so $n+1 \in X$. $\square$

Similarly, if $n \in X$ there is an e such that $W_e = \mathbb{N}$ and $e \in A$.

So if $n \in X$ and $\psi : F \to \Sigma$ extends ϕ, then $\psi(\omega) = \top$

Conversely, if $n \in X$, such an extension ψ exists.

What if $X = \emptyset$? Then $\phi(n) = \bot$. We can extend ϕ by $\psi(\omega) = \bot$.

Suppose $\{e \mid W_e = \mathbb{N}\} \subset A$ though. Pick e such that

$$e \cdot k \simeq \begin{cases} 0 & \text{if at stage } k,\ a \cdot e \text{ has not yet been computed} \\ \text{undefined} & \text{else} \end{cases}$$

If $e \notin A$, then $W_e = \mathbb{N}$ so $e \in A$; $\lightning$.

Hence $e \in A$, so $W_e = [0, \ldots, m-1]$ for some m; but then $m \in X$. $\lightning$

So ϕ extends to $\psi : F \to \Sigma$ in exactly one way.

2. A model in "modified realizability"

Now we consider 'Modified assemblies':

Objects are triples $(X, |\cdot|_X, P_X)$ such that $(X, |\cdot|_X)$ is an assembly, and $|x|_X \subset P_X \subset \mathbb{N}$, for all $x \in X$

Arrows are functions $f: X \to Y$ such that there is a partial recursive function φ satisfying:

i) $x \in X, n \in |x|_X \Rightarrow \varphi(n) \in |f(x)|_Y$

ii) $n \in P_X \Rightarrow \varphi(n) \in P_Y$

Modified assemblies are also the $\neg\neg$-separated objects of a topos. But in this topos, Markov's Principle fails.

Let

$$\Sigma = \{p \in \Omega \mid \exists e \in \mathbb{N}\, (p \leftrightarrow \neg\neg e \in K)\}$$

Σ corresponds to the modified assembly

$(\{\top, \bot\}, |\cdot|_\Sigma, \mathbb{N})$ $\quad |\top|_\Sigma = K \quad |\bot|_\Sigma = \overline{K}$

$F = (\omega+1, |\cdot|_F, \mathbb{N})$ $\quad |n|_F = \{e \mid W_e = [0,\ldots,n-1]\}$

$|\omega|_F = \{e \mid W_e = \mathbb{N}\}$

One can prove: for modified realizability

$I = \{\psi \in F : \exists n\, \neg\psi(n)\}$

And can be given by

$I = (\omega, |\cdot|_I, \mathbb{N})$

$|n|_I = \{\langle e, m\rangle \mid m \geq n \;\&\; W_e = [0,\ldots,n-1]\}$

Again, in Modified Assemblies, Σ is complete.

Yet there are differences with the situation for Assemblies:

ⓐ In Assemblies, complete $\Rightarrow$ well-complete

In Modified assemblies, $2 = 1+1$ is complete, but not well-complete

ⓑ In Assemblies, the Scott axiom holds:

(S) For all functions $P : \Sigma^N \to \Sigma$,

$$P(\lambda n.\top) \Rightarrow \exists n \in N.\, P(\hat{n})$$

(where $\hat{n} \in \Sigma^N$ is: $\hat{n}(k) = \begin{cases} \top & k < n \\ \bot & k \geq n \end{cases}$)

In Modified Assemblies, this fails

3. There are various models in Grothendieck toposes

Often, a subcategory C of ω-cpo is taken, such that C is <u>dense</u> in ω-cpo; with a subcanonical topology on C.

5. Other 'Synthetic' developments

Recently, two attempts have been made at 'synthesizing' mathematical theories. These are:

1. Escardó's 'synthetic topology'
2. Bauer's 'synthetic computability theory'.

Discuss some elements of 2.

Upshot: using ideas from synthetic domain theory, it seems possible to give a treatment of elementary recursion theory, without going into computation models (like Turing machines).

Specifically, a theory is presented, whose interpretation in the effective topos yields theorems of recursion theory.

Starting point: higher order logic with Axiom of Dependent Choices for $\mathbb{N}$ (that is: $\mathbb{N}$ is internally projective), and Markov's Principle:

$$\forall f : 2^{\mathbb{N}} \quad \neg\forall n\, f(n)=0 \longrightarrow \exists n\, f(n)=1$$

Define a countable set as a ~~quotient of~~ set A such that there is $\mathbb{N} \twoheadrightarrow A+1$
(if $A \twoheadrightarrow 1$, can assume $\mathbb{N} \twoheadrightarrow A$)

Easy: $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$ are not countable

Definition. $\Sigma = \{p : \exists f \in 2^{\mathbb{N}} (p \leftrightarrow \exists n\, f(n)=1)\}$

Note: $2^N \twoheadrightarrow \Sigma$ Also: $\top, \bot \in \Sigma$

Define: $U \subseteq X$ open if there is $f : X \to \Sigma$ such that $U = f^{-1}(\top)$

Proposition: Every open subset of N is countable

Proof: let $U \subset N$ open, classified by $f : N \to \Sigma$

By internal projectivity of N, there is $\bar{f} : N \to \cancel{\Sigma}\, 2^N$

such that

$$\begin{tikzcd} & 2^N \arrow[d] \\ N \arrow[ur, "\bar{f}"] \arrow[r, "f"'] & \Sigma \end{tikzcd}$$

commutes

Define $e : N \times N \twoheadrightarrow 1 + U$ by

$$e(\langle n, m\rangle) = \begin{cases} n & \text{if } \bar{f}(n)(m) = 1 \\ * & \text{else} \end{cases}$$

So, if $U \subset N$ open and inhabited, there is $N \twoheadrightarrow U$.
$(U \twoheadrightarrow 1)$

Interpretation: Every non empty r.e. set is the image of a recursive function

By internal projectivity of N, and $2^N \twoheadrightarrow \Sigma$,
we have $2^N \simeq (2^N)^N \twoheadrightarrow \Sigma^N$

This means: for $U \subset N$ open, there is $r : N \times N \to 2$
such that $n \in U \leftrightarrow \exists m \in N . r(m,n) = 1$

Interpretation: Every r.e. set is a projection of a recursive set

Define: $N_\bot = \{ U \subset N \text{ open} \mid \forall m, n\, (m \in U \wedge n \in U \to m = n) \}$

Then $N_\bot^N$ the set of partial maps $N \to N$ with open domain

Partial recursive functions

Now, we introduce an axiom.

Axiom Σ^N is countable.

Consequence:

Classical Logic fails, because:

1) Since $\Sigma^N \twoheadrightarrow \Sigma$, Σ is countable
2) Were Σ decidable, $\Sigma \cong 2$, hence $\Sigma^N \simeq 2^N$
   but Σ^N countable, 2^N not.

Theorem $N_\perp^{\,N}$ is countable

Consequence: there is enumeration $\varphi_{(-)} : N \twoheadrightarrow N_\perp^{\,N}$

Hence, acceptable Gödel numbering of partial recursive functions

Theorem [Phoa Principle]

For every $u : \Sigma \to \Sigma$, $x \in \Sigma$,

$$u(x) = (u(\perp) \vee x) \wedge u(\top)$$

Corollary: every map $\Sigma \to \Sigma$ is monotone

Corollary [Scott Axiom]: Every $f : \Sigma^N \to \Sigma$ is monotone and for $u \in \Sigma^N$,

$$f(u) = \top \implies \exists n_1, \ldots, n_k \in u .\ f(\{n_1, \ldots, n_k\}) = \top$$

Interpretation: Rice-Shapiro Theorem

⋮

Rice, Recursion Theorem, ...

Further reading

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