Ultracategories seminar # 6 : Exercises

For a total of 10 points. To be submitted by April 17, 2019.

1. On this sheet of paper, we say a category $\mathscr{C}$ is *filtered* if for each pair of objects C, D in $\mathscr{C}$ there is an object E with maps $C \to E$ and $D \to E$, and for parallel maps $f, g\colon C \rightrightarrows D$ there is an object E with a map $h\colon C \to E$ such that $hf = hg$. We say $\mathscr{C}$ is *cofiltered* if $\mathscr{C}^{\mathrm{op}}$ is filtered. Show that filtered colimits commute with finite limits in Set. (1 point)

2. Recall that functors $F\colon \mathscr{C} \to \mathrm{Set}$ are naturally isomorphic to a colimit of representables.

   (a) Let $\mathscr{C}$ be a category with finite limits. Show a functor $F\colon \mathscr{C} \to \mathrm{Set}$ is naturally isomorphic to a filtered colimit of representables if and only if F preserves finite limits. (2 points)

   (b) Show the Yoneda embedding $\mathscr{C} \to \mathrm{Pro}(\mathscr{C})$ commutes with finite limits and all small colimits that exist in $\mathscr{C}$. (1 point)

3. Let Stone be the category of Stone spaces (and continuous maps), Bool the category of Boolean algebras, Fin the category of finite sets, and Fin Bool the category of finite Boolean algebras.

   (a) Show that every Boolean algebra is a filtered colimit of finite Boolean algebras. (1 point)

   Let $\mathscr{C}$ be a category with finite colimits. Define $\mathrm{Ind}(\mathscr{C}) = \mathrm{Pro}(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$. In part (a) we have essentially shown that $\mathrm{Ind}(\mathrm{Fin\,Bool}) \simeq \mathrm{Bool}$.

   (b) Use Stone duality to show that $\mathrm{Pro}(\mathrm{Fin}) \simeq \mathrm{Stone}$. (2 points)

   (c) Let $f\colon X \to Y$ be a continuous map of Stone spaces. Show that f is a homeomorphism if and only if for all finite sets S, the induced map

   $$\mathrm{Hom}_{\mathrm{Top}}(Y, S) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{Top}}(X, S)$$

   is a bijection. (1 point)

4. Let $\mathscr{C}$ be a pretopos. Show the functor of Construction 6.4.3 of the paper, defined by $g\colon \mathrm{Fin} \to \mathscr{C}$ sending $S \mapsto \coprod_S \mathbf{1}$, where $\mathbf{1}$ is the terminal object of $\mathscr{C}$, is left-exact. (2 points)