Seminar on Logic - 2018/2019. Exercise of the 17th of April.

We said that, since $y^{op}\colon \mathcal{C}^{op} \hookrightarrow \mathrm{Pro}(\mathcal{C})^{op}$ *is the free small filtered cocompletion of* $\mathcal{C}^{op}$*, there is a bijection between the class of continuous presheaves* $\mathrm{Pro}(\mathcal{C})^{op} \to \textsc{Set}$ *and the class of presheaves* $\mathcal{C}^{op} \to \textsc{Set}$*. This bijection is the precomposition with* y^{op}*.*

(a) - (3 points) Prove that this bijection is actually an equivalence of categories, that is, it is fully faithful.

A clearer proof of the fact that the precomposition with:

$$\Gamma\colon \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C})$$

induces an equivalence of categories $\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}^{cont}(\mathrm{Stone}_{\mathcal{C}})$*.*

Let $\mathcal{C}$ be a small pretopos.

(b) - (4 points) Without using that $\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ is a basis for the coherent topology over $\mathrm{Pro}(\mathcal{C})$ (as we did during the seminar), prove that the precomposition with the fully faithful functor:

$$\mathrm{Pro}^{wp}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$$

is an equivalence $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}^{wp}(\mathcal{C}))$, exhibiting its pseudo-inverse. *Hint: use Theorem 6.2.12 and look into the proof of Corollary 7.2.4.*

(c) - (3 points) Prove that this equivalence restrics to an equivalence:

$$\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}^{cont}(\mathrm{Pro}^{wp}(\mathcal{C}))$$

and so conclude the usual equivalence:

$$\mathrm{Shv}^{cont}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}^{cont}(\mathrm{Stone}_{\mathcal{C}})$$

induced by $\Gamma\colon \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C})$.