(a) Suppose that we have an object $(X, \mathcal{O}_X)$ of $\mathrm{Comp}_{\mathcal{M}}$ and, for each $t \in T$, a map $\underline{M_t} \to (X, \mathcal{O}_X)$. By Remark 4.2.6, such a map is determined by an element $x_t \in X$ and an arrow $g_t \colon \mathcal{O}_{X,x_t} \to M_t$. We need to show that there exists a unique arrow $(f, \alpha) \colon (\beta T, \mathcal{O}_{\beta T}) \to (X, \mathcal{O}_X)$ such that the diagram

$$\begin{array}{ccc} \underline{M_t} & & \\ \downarrow & \searrow & \\ (\beta T, \mathcal{O}_{\beta T}) & \xrightarrow[(f,\alpha)]{} & (X, \mathcal{O}_X) \end{array}$$

commutes for all $t \in T$. The commutativity of this diagram comes down to the following two requirements:

(i) $f(\delta_t) = x_t$;

(ii) $\varepsilon_{T,t} \circ \alpha_{\delta_t} = g_t$.

By the universal property of βT (Proposition 3.2.7), there exists a unique continuous map $f \colon \beta T \to X$ satisfying (i) for all $t \in T$. By Proposition 4.2.9, there exists a unique natural transformation of left ultrafunctors $\alpha \colon \mathcal{O}_X \circ f \to \mathcal{O}_{\beta T}$ satisfying (ii) for all $t \in T$. This completes the proof. $\square$

(b) It is given in the exercise that u_* is continuous. So, in order to prove that (u_*, α) is a morphism, it remains to show that α is a natural transformation of left ultrafunctors. That is, we need to show that α is compatible with the left ultrastructures σ_μ described in Proposition 4.2.8. Explicitly, let $\nu_\bullet \colon S \to \beta T$ be a map of sets, and let $\mu \in \beta S$. We need to show that

$$\begin{array}{ccccc} \mathcal{O}_{\beta T, u_*(\int_S \nu_s d\mu)} & = & \mathcal{O}_{\beta T, \int_S u_* \nu_s d\mu} & \xrightarrow{\sigma_\mu} & \int_S \mathcal{O}_{\beta T, u_* \nu_s} d\mu \\ \downarrow {\scriptstyle \alpha_{\int_S \nu_s d\mu}} & & & & \downarrow {\scriptstyle \int_S \alpha_{\nu_s} d\mu} \\ \mathcal{O}_{\beta T_0, \int_S \nu_s d\mu} & & \xrightarrow[\sigma_\mu]{\qquad\qquad} & & \int_S \mathcal{O}_{\beta T_0, \nu_s} d\mu \end{array}$$

commutes. We obtain this diagram by appending the two squares

$$\begin{array}{ccc} \int_T M_t d\left(\int_{T_0} \delta_{t_0} d\left(\int_S \nu_s d\mu\right)\right) & = & \int_T M_t d\left(\int_S \left(\int_{T_0} \delta_{t_0} d\nu_s\right) d\mu\right) \\ & & \downarrow {\scriptstyle \Delta_{\mu, \int_{T_0} \delta_{t_0} d\nu_\bullet}} \\ \downarrow {\scriptstyle \Delta_{\int_S \nu_s d\mu, \delta_\bullet}} & & \int_S \left(\int_T M_t d\left(\int_{T_0} \delta_{t_0} d\nu_s\right)\right) d\mu \\ & & \downarrow {\scriptstyle \int_S \Delta_{\nu_s, \delta_\bullet} d\mu} \\ \int_{T_0} \left(\int_T M_t d\delta_{t_0}\right) d\left(\int_S \nu_s d\mu\right) & \xrightarrow{\Delta_{\mu, \nu_\bullet}} & \int_S \left(\int_{T_0} \left(\int_T M_t d\delta_{t_0}\right) d\nu_s\right) d\mu \\ \downarrow {\scriptstyle \int_{T_0} \varepsilon_{T,t_0} d\left(\int_S \nu_s d\mu\right)} & & \downarrow {\scriptstyle \int_S \left(\int_{T_0} \varepsilon_{T,t_0} d\nu_s\right) d\mu} \\ \int_{T_0} M_{t_0} d\left(\int_S \nu_s d\mu\right) & \xrightarrow{\Delta_{\mu, \nu_\bullet}} & \int_S \left(\int_{T_0} M_{t_0} d\nu_s\right) d\mu \end{array}$$

where the top square commutes by axiom (C) of an ultracategory, the bottom square commutes by the naturality of $\Delta_{\mu,\nu_\bullet}$ (axiom (3)), and $\int_S \left(\int_{T_0} \varepsilon_{T,t_0} d\nu_s \right) d\mu \circ \int_S \Delta_{\nu_s,\delta_\bullet} d\mu$ is equal to $\int_S \alpha_{\nu_s} d\mu$ by the functoriality of $\int_S \bullet d\mu$ (axiom (1)). Finally, (u_*, α) is cartesian since $\alpha_\nu = \Delta_{\nu,u}$ is an isomorphism for each $\nu \in \beta T_0$, by axiom (B). □

Remark: everyone forgot to mention the functoriality of $\int \bullet d\mu$.

(c) We obtain this diagram by appending the diagrams

$$\begin{array}{ccc}
\int_T M_t d\left(\int_{T_0} \delta_{t'_0} d\delta_{t_0}\right) & = & \int_T M_t d\delta_{t_0} \\
\Big\downarrow {\scriptstyle \Delta_{\delta_{t_0},\delta_\bullet}} & & \Big\downarrow {\scriptstyle \mathrm{id}} \\
\int_{T_0} \left(\int_T M_t d\delta_{t'_0}\right) d\delta_{t_0} & \xrightarrow{\varepsilon_{T_0,t_0}} & \int_T M_t d\delta_{t_0} \\
\Big\downarrow {\scriptstyle \int_{T_0} \varepsilon_{T,t'_0} d\delta_{t_0}} & & \Big\downarrow {\scriptstyle \varepsilon_{T,t_0}} \\
\int_{T_0} M_{t'_0} d\delta_{t_0} & \xrightarrow{\varepsilon_{T_0,t_0}} & M_{t_0}
\end{array}$$

where the top square commutes by axiom (A) and the bottom square commutes by the naturality of ε_{T_0,t_0} (axiom (2)). □

Remark: some of you claimed that $\Delta_{\delta_{t_0},\delta_\bullet}$ is the inverse of $\varepsilon_{T_0,t_0} \colon \int_{T_0} M_{t'_0} d\delta_{t_0} \to M_{t_0}$. This is not correct, however, since the domain of $\Delta_{\delta_{t_0},\delta_\bullet}$ is not a map $M_{t_0} \to \int_{T_0} M_{t'_0} d\delta_{t_0}$.

(d) By exercise (a), the canonical map $\bigsqcup_{t_0 \in T_0} \underline{M_{t_0}} \to \bigsqcup_{t \in T} \underline{M_t}$ is $(f, \alpha') \colon (\beta T_0, \mathcal{O}_{\beta T_0}) \to (\beta T, \mathcal{O}_{\beta T})$, where:

(i) f is the unique continuous map $\beta T_0 \to \beta T$ such that $f(\delta_{t_0}) = \delta_{t_0}$ for all $t_0 \in T_0$;

(ii) α' is the unique natural transformation of left ultrafunctors $\mathcal{O}_{\beta T} \circ f \to \mathcal{O}_{\beta T_0}$ such that $\varepsilon_{T_0,t_0} \circ \alpha'_{\delta_{t_0}} = \varepsilon_{T,t_0}$ for every $t_0 \in T_0$.

By the remark preceding exercise (c), we must have $f = u_*$. By exercises (b) and (c), we get $\alpha' = \alpha$, which completes the proof. □

Marking scheme

(a) 1pt Spelling out the conditions (i) and (ii) that (f, α) needs to satisfy.

1pt Using Proposition 3.2.7 to deduce that f is uniquely determined.

1pt Using Proposition 4.2.9 to deduce that α is uniquely determined.

(b) $\frac{1}{2}$pt Formulating the diagram that needs to commute (either the first diagram in the solution, or already spelled out in terms of Δ and ϵ).

2pt Appending the right diagrams in order to obtain the desired diagram. One should mention the axioms that are being used (in this case: (C), (3) and (1)). Failing to mention these in some way leads to a $\frac{1}{2}$pt subtraction (per axiom, up to a maximum of 1pt).

$\frac{1}{2}$pt Mentioning that (u_*, α) is cartesian by axiom (B).

(c) 2pt Appending the right diagrams in order to obtain the desired diagram. One should mention the axioms that are being used (in this case: (A) and (2)). Failing to mention these in some way leads to a $\frac{1}{2}$pt subtraction

(d) 1pt Showing that the underlying continuous function is u_*.

1pt Showing that the natural transformation of left ultrafunctors is α.