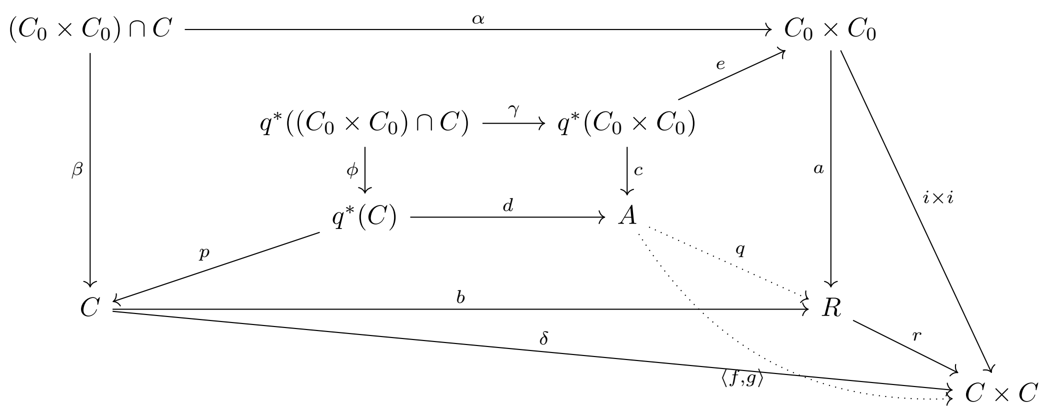
Seminar on Logic. Exercise to be handed in on 15/05/19

1. We will prove that the relation $\{(f,g)|\langle f,g\rangle : Y \to C\times C$ factors through $R\}$ is an equivalence relation for every object $Y \in C$. We will show the three characteristics.

   For reflexivity, we look at an arbitrary arrow $f : Y \to C$. We want to see whether $\langle f,f\rangle$ factors through R. This is easy to see, since $\delta f = \langle f,f\rangle$, and since R is the join of δ with something else, we see that δ factors through R. So $\langle f,f\rangle$ also factors through R.

   For symmetry, suppose that $\langle f,g\rangle$ factors through R as rq, where $r : R \to C\times C$ is the map exhibiting the equivalence relation. Since the formation of unions and intersections is compatible with pullback, we can look at the following diagram, where both the inner and outer squares are both a pullback and a pushout:

   By definition and construction of all the given maps, we see that every part of this diagram commutes. So in particular we see that $\delta p\phi = (i\times i)e\gamma$ and $\langle f,g\rangle$ is the unique arrow that exhibits the inner square as a pushout. Now since δ and $i\times i$ are maps that are invariant under the twist map, we have that $\delta p\phi = (i\times i)\text{tw}(e)\gamma$ and $\text{tw}(\langle f,g\rangle) = \langle g,f\rangle$ is the unique map exhibiting the inner square as a pushout.

   But now we see the following. By some diagram chasing, we see that $rbp\phi = ra\text{tw}(e)\gamma$, and since r is mono, $bp\phi = a\text{tw}(e)\gamma$. This means that there is a unique arrow $y : A \to R$ that makes everything commute. But then we have that $\delta p\phi = rbp\phi = ryd\phi = ryc\gamma = (i\times i)\text{tw}(e)\gamma$. But we concluded earlier that $\langle g,f\rangle$ was the unique arrow with these properties. So we may conclude that $\langle g,f\rangle = ry$ and therefore $\langle g,f\rangle$ factors through R.

   Now for the last part, I must admit that I somewhat underestimated the amount of diagram chasing that this would involve. As seen with the symmetry, the diagram would become even bigger, but the argument not necessarily harder. A sketch of the proof is as follows. Suppose we have that $\langle f,g\rangle$ and $\langle g,h\rangle$ factor through R as q_1r and q_2r respectively. We take pullbacks along q_1 and q_2 like in the symmetry case. Now we can also see that $i\times i$ and δ are transitive, in the sense that they behave well with respect to both the induced pullback squares. And from that we have with a similar pushout-like argument as above, we see that $\langle f,h\rangle$ factors through R. Drawing out the entire diagram is quite tedious, and the details will therefore be omitted.

2. The idea is to transform the conditions given by (iv) into a commutative diagram like with (iii), and see whether the map that (iii) gives us tells us that the sheaf is actually lifted.

   By the Yoneda lemma, we have that global sections s_X, regarded as elements of $\mathcal{O}_X^C(X)$, are in natural correspondence with arrows $\Gamma(X,\mathcal{O}_X) \to \text{Hom}(C,(-))$. Now we also see by the Yoneda Lemma that $f(s_Y) \in \mathcal{O}_X^C(X)$ corresponds to the map Γf composed with the map $\Gamma(X,\mathcal{O}_X) \to \text{Hom}(C,(-))$ induced by s_X, which we denote by x.

Now the fact that s_X lifts to a section in the subsheaf $\mathcal{O}_X^{C_0}(X)$ means that the map $x\Gamma f$ factors through the inclusion map $i : C_0 \to C$, or rather the induced inclusion map between Hom-functors, but we denote this the same way. So in particular we must have that the following square commutes:

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \longrightarrow & C_0 \\ \downarrow & & \downarrow \\ \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & C \end{array}$$

Now if we look at (iii) and we fill in that $P = \Gamma(X, \mathcal{O}_X)$ and $Q = \Gamma(Y, \mathcal{O}_Y)$ we may conclude that there exists a map $\Gamma(Y, \mathcal{O}_Y) \to C_0$ which makes the diagram commute. And by similar reasoning as above, this means that s_Y lifts to a section of $\mathcal{O}_Y^{C_0}$. And that is exactly what (iv) says.

3. (a) First we check the regular definition of a compatible family. Let R be a sieve on $(X, \mathcal{O}(X))$. We see that a compatible family is a collection of elements $(x_f | f \in R)$ such that $x_f \in \mathscr{F}(dom(f))$ such that for all $f \in R$ and arbitrary g with $ran(g) = dom(f)$ we have that $x_{fg} = \mathscr{F}(g)(x_f)$. Now suppose that we have such a compatible family. We look at the following commuting diagram where f_i and f_j are elements of R and h and g are arbitrary:

$$\begin{array}{ccc} (W, \mathcal{O}_W) & \xrightarrow{h} & (X_i, \mathcal{O}_{X_i}) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_i} \\ (X_j, \mathcal{O}_{X_j}) & \xrightarrow{f_j} & (X, \mathcal{O}_X) \end{array}$$

We see that $f_j g = f_i h$ so in particular $\mathscr{F}(g)(x_{f_j}) = x_{f_j g} = x_{f_i h} = \mathscr{F}(h)(x_{f_j})$. So then indeed we have that x_{f_j} and x_{f_i} have the same image.

For the converse, assume that the x_{f_j} and x_{f_i} have the same image for all arrows in the sieve. We want to show that for any arrow g we have that $x_{f_i g} = \mathscr{F}(g)(x_{f_i})$. But we have this pretty much immediately, since $f_i G$ is an element of R and therefore we can look at the following commuting diagram:

$$\begin{array}{ccc} (X_j, \mathcal{O}_{X_j}) & \xrightarrow{g} & (X_i, \mathcal{O}_{X_i}) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f_i} \\ (X_j, \mathcal{O}_{X_j}) & \xrightarrow{f_i g} & (X, \mathcal{O}_X) \end{array}$$

Now by assumption we have that $x_{f_i g} = \mathscr{F}(\mathrm{id})(x_{f_i g}) = \mathscr{F}(g)(x_{f_i})$.

(b) The easiest way to solve this exercise is the see that amalgamation is uniquely characterized by the fact that an amalgamation is a point that is sent to every of the x_f by the maps $\mathscr{F}(f)$. So if we have that x is an amalgamation, so $x_f = \mathscr{F}(f)(x)$, then we have that $x_{fg} = \mathscr{F}(fg)(x) = (F)(g)(x_f)$. This implies that the induced map from the question is injective, since this map can be seen as sending amalgamations to their compatible families.

(c) By a similar argument as above, we see that the uniqueness of the amalgamation is exactly equivalent to the fact that there exists a point x with the given properties.

4. The following is a list of possible typos. There are probably more to be found.

- Page 91 in the proof of 7.4.3, the definition of θ should be $M(D_0) \to N(D_0) \times_{N(D)} M(D)$.
- In the statement of Lemma 7.5.4. the ultrapower diagonal should be denoted as Δ_μ, not as δ_M.
- In the last line of page 96, it says $\mathscr{F}_0(\mathcal{O}_{X_i,y})$. This should be $F(\mathcal{O}_{X_i,y})$.