Seminar Ultracategories Exercise 9: model solution

Tim Baanen

In the presentation, we discussed categories with unique factorisation: if a category has small products and all objects are products of coconnected objects, then this product of coconnected objects is essentially unique. In this hand-in exercise, we will study the converse: if a category has essentially unique factorisation into some subcategory of factors, are these factors always coconnected?

1. (3pt) Fix a category $\mathcal{E}$ with all small products and finite colimits, such that $\mathcal{E}^{\mathsf{op}}$ is extensive. In particular $\mathcal{E}$ satisfies: projections from finite products are epimorphisms and pushforwards preserve products. Let $X \in \mathcal{E}$ be an object such that $X \simeq U \times V$ implies that exactly one of U and V is terminal. Show that $\mathrm{Hom}(\_, X)$ sends binary products to disjoint unions.

We want to show for all $f : Y \times Z \to X$ that there is a unique $f_0 : Y \to X$, or $f_1 : Z \to X$ such that $f_0 \circ \pi_Y = f$ or $f_1 \circ \pi_Z = f$, i.e. $\mathrm{Hom}(\_, X)$ preserves binary products. Let $f \in \mathrm{Hom}(Y \times Z, X)$ for arbitrary Y, Z, and define $u : X \to U$ to be the pushforward of $\pi_Y : Y \times Z \to Y$ along f, and $v : X \to V$ be the pushforward of $\pi_Z : Y \times Z \to Z$ along f, as in the following diagrams:

$$\begin{tikzcd} Y \times Z \arrow[r, "f"] \arrow[d, "\pi_Y"] & X \arrow[d, "u"] \\ Y \arrow[r, "f_0"] \arrow[ur, phantom, "\ulcorner", very near end] & U \end{tikzcd} \qquad \begin{tikzcd} Y \times Z \arrow[r, "f"] \arrow[d, "\pi_Z"] & X \arrow[d, "v"] \\ Z \arrow[r, "f_1"] \arrow[ur, phantom, "\ulcorner", very near end] & V. \end{tikzcd}$$

Since pushforwards preserve products, we have that $X \simeq (Y \times Z) +_{Y \times Z} X \simeq (Y +_{Y \times Z} X) \times (Z +_{Y \times Z} X) = U \times V$. Now we have WLOG that $V \simeq 1$, and $U \simeq X$, so f factors as $f_0 \circ \pi_Y$.

Moreover, f_0 is the unique map such that composing with π_Y gives f, since π_Y is epi by assumption. If f factors as $f_1 \circ \pi_Z$ for $f_1 : Z \to X$, then we have that a pushout of f along π_Z is given by f_1, so $X \simeq V \simeq 1$. This implies that $X \simeq U \simeq 1$. Thus, we have that f factors only through one of π_Y or π_Z.

Grading: **1pt** for U and V as pushforward along f, **1pt** for finding f_0, **1pt** for showing f_0 is unique.

2. (2pt) Suppose $\mathcal{E}$ as in the first exercise has a full subcategory $\mathcal{C}$ such that each $X \in \mathcal{E}$ is an essentially unique product of objects in $\mathcal{C}$.[1] Show that $\mathcal{E}$ is a category with unique factorisation, and in particular $\mathcal{C} \subseteq \mathcal{E}^{\mathsf{cc}}$.

Let $X \in \mathcal{C}$, $U, V \in \mathcal{E}$ such that $X \simeq U \times V$. Factor U as $U \simeq \prod_{i \in I} C_i$ and V as $V \simeq \prod_{j \in J} C_j$ for $C_i, C_j \in \mathcal{C}$, so that we get $X \simeq \prod_{i \in I \sqcup J} C_i$. Note that we can factor X essentially uniquely as the unary product of X itself, so we have that $I \sqcup J$ has exactly one object. Thus, exactly one of I and J is empty, so exactly one of U and V is a terminal object.

We want to use this to prove all $X \in \mathcal{C}$ are coconnected, i.e. $\mathrm{Hom}(\_, X)$ send finite products to disjoint unions. It remains to show $\mathrm{Hom}(1, X) = \varnothing$ for all such X since the first exercise takes care of binary products. Moreover, because $\mathrm{Hom}(\_, X)$ sends binary products to disjoint unions and $1 \times X \simeq X$, we have $\mathrm{Hom}(1, X) \sqcup \mathrm{Hom}(X, X) \simeq \mathrm{Hom}(1 \times X, X) \simeq \mathrm{Hom}(X, X)$. In the second alternative, this isomorphism restricts to $\mathsf{id} : \mathrm{Hom}(X, X) \to \mathrm{Hom}(X, X)$, so we must have that the first alternative, $\mathrm{Hom}(1, X)$ is empty. In other words, $\mathrm{Hom}(\_, X)$ sends terminal objects to the initial object. Together with preserving binary products, this gives that $\mathrm{Hom}(\_, X)$ sends all finite products in $\mathcal{E}$ to coproducts in Set, i.e. X is coconnected.

[1] That is: if $X \simeq \prod_{s \in S} X_s \simeq \prod_{t \in T} X'_t$ for $X_s, X'_t \in \mathcal{C}$ then there is a bijection $\phi : S \simeq T$ and isomorphisms $X_s \simeq X'_{\phi(s)}$.

We assumed that $\mathcal{E}$ has all small products and each object in $\mathcal{E}$ is an essentially unique product of objects in $\mathcal{C}$, which implies that each object in $\mathcal{E}$ is a product of coconnected objects, since we have $\mathcal{C} \subseteq \mathcal{E}^{\mathsf{cc}}$. Now we already assumed $\mathcal{E}$ has all small products, so $\mathcal{E}$ is a unique factorisation category.

Grading: **1pt** for applying the first exercise (verifying its conditions), **1pt** for checking the terminal object.

3. (3pt) Let Y be a compact Hausdorff space, which we will view as a discrete ultracategory. Show that $\mathrm{Env}(Y)$ has has all small products, finite colimits and that $\mathrm{Env}(Y)^{\mathsf{op}}$ is extensive.
   Hint: first prove that $\mathrm{Env}(Y) \subseteq \mathsf{Stone}_Y^{\mathsf{op}}$ *is equivalent to the opposite of the slice category,* $(\mathsf{Set}/Y)^{\mathsf{op}}$.

Since $\mathrm{Env}(Y)$ is constructed to be an envelope of Y, it has all small products, so it remains to show that $\mathrm{Env}(Y)$ has all finite colimits and that $\mathrm{Env}(Y)^{\mathsf{op}}$ is extensive, or after we prove the hint, that Set/Y has all finite limits and is extensive. Still, we will explicitly show that Set/Y also has all small coproducts. To show the equivalence of the hint, there are multiple approaches:

1. Here, the first step is showing that the larger category Stone_Y is equivalent to Stone/Y. (See also Example 4.1.4.) The objects of $\mathsf{Stone}_Y^{\mathsf{op}}$ are Stone spaces $X \in \mathsf{Stone}$ with a left ultrafunctor $f \in \mathrm{Fun}^{\mathrm{LUlt}}(X, Y)$, and left ultrafunctors between compact Hausdorff spaces are just continuous functions. The morphisms from $(X, f) \to (X', f')$ in Stone_Y are continuous maps $g : X \to X'$, together with a natural transformation of left ultrafunctors $\alpha : f' \circ g \Rightarrow f$, and since the categories are discrete, these are just equalities between the two continuous maps $f' \circ g = f$. We can see that writing out the definition of Stone_Y gives exactly the definition of Stone/Y. In this equivalence, an object of $\mathrm{Env}(Y)$ (which is defined to be a product of coconnected objects) corresponds to a coproduct in Stone/Y of a family of constant maps $\{\mathsf{const}(y_s) : \{s\} \to Y\}_{s \in S}$. Coproducts in the slice category Stone/Y are computed by taking the coproduct of the underlying space in Stone and giving the unique map from this coproduct to Y, and the coproduct of one-point spaces are exactly the discrete spaces, i.e. just all sets. We conclude that the objects of $\mathrm{Env}(Y)$ correspond dually to sets S together with a map $f : S \to Y$ to the underlying set of the space Y, and since we take the full subcategory of $(\mathsf{Stone}/Y)^{\mathsf{op}}$ of discrete spaces, we conclude $\mathrm{Env}(Y)$ is equivalent to $(\mathsf{Set}/Y)^{\mathsf{op}}$.

2. An alternative approach is based on Example 8.4.2: we show $\mathrm{Env}(Y)$ is the subcategory of $\mathsf{Stone}_Y^{\mathsf{op}}$ spanned by $(\beta S, \mathcal{O}\beta S)$. Then we have that the spaces βS are dual, via Stone duality, to complete atomic boolean algebras, which are themselves dual to sets via the powerset functor. Composing gives an equivalence that sends Stone spaces βS to sets S and continuous maps $\beta S \to Y$ into functions of sets $S \to Y$, so applying this to the objects of the form $(\beta S, \mathcal{O}\beta S)$ in $\mathsf{Stone}_Y^{\mathsf{op}}$ gives the objects of $(\mathsf{Set}/Y)^{\mathsf{op}}$.

3. Finally, we can consider the functor $\beta : \mathsf{Set} \to \mathsf{Stone}$, and prove it extends to a full and faithful functor $F : \mathsf{Set}/Y \to \mathsf{Stone}_Y$. Its essential image consists of objects of the form $(\beta S, \mathcal{O}\beta S)$, which again by Example 8.4.2 is $\mathrm{Env}(Y)^{\mathsf{op}}$.

Now it remains to verify Set/Y has all finite limits and small coproducts, and is extensive. Since Set is a topos, and the fundamental theorem of topos theory states that slices of toposes are themselves toposes, we have that Set/Y is a topos, and in particular extensive and has all colimits. More explicitly: the terminal object is given by $\mathsf{id} : Y \to Y$, and the product in Set/Y of $f : X \to Y$ and $f' : X' \to Y$ is the pullback in Set of f and f', while pullbacks and coproducts in Set/Y are given by pullbacks and coproducts respectively in Set, making use of the unique mapping property of (co)limits. Extensivity of the slice category is then exactly the same condition as extensivity in Set. Thus, the equivalence $\mathrm{Env}(Y) \cong (\mathsf{Set}/Y)^{\mathsf{op}}$, together with the result that Set/Y is extensive and has all finite limits and small coproducts, shows that $\mathrm{Env}(Y)$ satisfies the required conditions.

Grading: **2pt** for $\mathrm{Env}(Y) \cong (\mathsf{Set}/Y)^{\mathsf{op}}$ (through one of the various methods), **1pt** for extensivity and finite limits in Set/Y.

4. (2pt) A partial counterexample: let the dual factor lattice $\mathsf{F}_\omega^{\mathsf{op}}$ be the category with an object for each positive natural number $n \in \mathbb{N}_{>0}$, and a unique arrow $c : a \to b$ if there is a $c \in \mathbb{N}$ with $a = bc$. Show that there is a subcategory $\mathcal{C} \subseteq \mathsf{F}_\omega^{\mathsf{op}}$ such that each object in $\mathsf{F}_\omega^{\mathsf{op}}$ is an essentially unique product[2] of objects in $\mathcal{C}$, but the objects of $\mathcal{C}$ are *not* all coconnected.

[2]Warning: the product in $\mathsf{F}_\omega^{\mathsf{op}}$ is not necessarily given by multiplication in $\mathbb{N}$!

The categorical binary product $a \times b$ for $a, b \in \mathsf{F}_\omega^{\mathsf{op}}$ is the least common multiple $\mathrm{lcm}(a, b)$, since $\mathrm{lcm}(b, c) \mid a$ if and only if $b \mid a$ and $c \mid a$.

Note that 1 is the terminal object, so it is the empty product, and moreover this factorisation is unique since positive prime powers do not divide 1, so there is no arrow from 1 into an element of P^*.

Moreover, we cannot uniquely factor objects in $\mathsf{F}_\omega^{\mathsf{op}}$ as coconnected objects, since no objects in $\mathsf{F}_\omega^{\mathsf{op}}$ are coconnected. More precisely, for all $n \in \mathsf{F}_\omega^{\mathsf{op}}$ we have that $n \times n = \mathrm{lcm}(n, n) = n$, and $\mathrm{Hom}(n \times n, n) = \mathrm{Hom}(n, n) \not\cong \mathrm{Hom}(n, n) \sqcup \mathrm{Hom}(n, n)$. Thus, we cannot write objects in $\mathsf{F}_\omega^{\mathsf{op}}$ as products of coconnected objects.

Unfortunately, at this point I did not realize that this means we can change the number of copies of prime powers appearing in the prime factorisation: since $\mathrm{lcm}(p, p) = p$ we can factor each prime as p or as $p \times p$. The only arrows going into p are the identity from p itself, so the only object in $\mathsf{F}_\omega^{\mathsf{op}}$ with unique factorisation in $\mathsf{F}_\omega^{\mathsf{op}}$ is 1.

I thought that the following would work: Let P^* be the set of positive prime powers: $P^* = \{p^n \mid p \text{ is prime and } n > 0\}$, and we want to show that P^* satisfies the conditions of $\mathcal{C}$. The fundamental theorem of arithmetic states that each natural number $n > 1$ can be written uniquely as a product of (finitely many) prime powers up to permutation of the factors. This permutation is the bijection ϕ between the indices, while the isomorphisms between the factors are just the identity maps. But we do not have to factor n into distinct prime powers if we take the least common multiple, so this argument fails.

Grading: **1pt** for noticing the flaw in the exercise, **1pt** for showing that $\mathcal{C}$ cannot be all coconnected.