Hand-in 6

Course: Seminar Logic - Categorical Logic

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This hand-in consists of three exercises.

Exercise 1. (3 + 7 points) In this exercise we consider the category Top of topological spaces and continuous function. We make it into a prop-category by taking for X a topological space, $\mathrm{Prop}_{\mathrm{Top}}(X)$ to be the topology on X ordered by inclusion and for $f : Y \to X$ a continuous function, the pullback operation $f^* : \mathrm{Prop}_{\mathrm{Top}}(X) \to \mathrm{Prop}_{\mathrm{Top}}(Y)$ to be defined by taking the preimage through f. This prop-category has binary meets given by taking the intersection and for each topological space X a top element $\top_X = X$ of $\mathrm{Prop}_{\mathrm{Top}}(X)$. Both are preserved by the pullback operations such that in conclusion Top has finite meets. We will now study if Top has universal quantifiers.

a. Show that for I, X topological spaces the pullback operation $\pi_1^* : \mathrm{Prop}_{\mathrm{Top}}(I) \to \mathrm{Prop}_{\mathrm{Top}}(I \times X)$ induced by the projection $\pi_1 : I \times X \to I$ has a right adjoint $\bigwedge_{I,X} : \mathrm{Prop}_{\mathrm{Top}}(I \times X) \to \mathrm{Prop}_{\mathrm{Top}}(I)$ and describe it explicitly.

Solution. Inspired by the case for Set we define for some open $A \subseteq I \times X$ the open $\bigwedge_{I,X}(A) \subseteq I$ to be the interior of $\{i \in I | \forall x \in X((i,x) \in A)\}$. We show that this gives the adjoint of $\pi_1^* : \mathrm{Prop}_{\mathrm{Top}}(I) \to \mathrm{Prop}_{\mathrm{Top}}(I \times X)$. Let $A \in \mathrm{prop}_{\mathrm{Top}}(I), B \in \mathrm{prop}_{\mathrm{Top}}(I \times X)$. Suppose that $A \subseteq \bigwedge_{I,X}(B)$. We need to show $\pi_1^*(A) \subseteq B$. Let $(i,x) \in \pi_1^*(A)$. Then

$$i = \pi_1(i,x) \in A \subseteq \bigwedge_{I,X}(B) \subseteq \{i \in I | \forall x \in X((i,x) \in B)\}.$$

It follows that $(i,x) \in B$. We conclude that $\pi_1^*(A) \subseteq B$.

Suppose conversely that $\pi_1^*(A) \subseteq B$. We need to show that $A \subseteq \bigwedge_{I,X}(B)$. Let $i \in A$ and let $x \in X$. Then $\pi_1^*(i,x) = i \in A$ so $(i,x) \in \pi_1^*(A) \subseteq B$. We see $i \in \{j \in I | \forall x \in X((i,x) \in B)\}$ and more general $A \subseteq \{j \in I | \forall x \in X((i,x) \in B)\}$. By definition of $\bigwedge_{I,X}$ we now get that $A \subseteq \bigwedge_{I,X}(B)$. We conclude that

$$A \subseteq \bigwedge_{I,X}(B) \Leftrightarrow \pi_1^*(A) \subseteq B$$

such that $\bigwedge_{I,X}$ is the adjoint of π_1^*. △

b. Show that Top does not have universal quantifiers. *Hint. Suppose for a contradiction that* Top *does have universal quantifiers and consider the case where $I = \mathbb{R}_{\geq 0}$ with the regular topology and $X = \mathbb{R}_{>0}$ with the discrete topology.*

Solution. Suppose for a contradiction that Top has universal quantifiers. Then for any continuous $f : I' \to I$ and topological space X we have the naturality condition:

$$f^*\left(\bigwedge_{I,A}(A)\right) = \bigwedge_{I',X}((f \times 1_X)^*(A))$$

for each $A \in \text{prop}_{\text{Top}}(I \times X)$.

Let now I, X be topological spaces and $A \in \text{prop}_{\text{Top}}(I \times X)$. We will shows $\bigwedge_{I,X}(A) = \{j \in I | \forall x \in X((i,x) \in A)\}$. The inclusion $\bigwedge_{I,X}(A) \subseteq \{j \in I | \forall x \in X((i,x) \in A)\}$ is a direct consequence of the definition of $\bigwedge_{I,X}$ so it remains to prove the inverse inclusion. Let $i \in \{j \in I | \forall x \in X((i,x) \in A)\}$. We take $I' = \{*\}$. Then there is unique continuous function $f : I' \to I$ sending $*$ to i and for each $x \in X$ we have $(f \times 1_X)(*, x) = (i, x) \in A$. We see that for each $x \in X$ we have $(*, x) \in (f \times 1_X)^*(A)$. Consequently we have

$$I' = \{*\} \subseteq \{j \in I' | \forall x \in X((j,x) \in (f \times 1_X)^*(A))\} \subseteq I'$$

so $\{j \in I' | \forall x \in X((i,x) \in (f \times 1_X)^*(A)))\} = I'$. As I' is open in itself we have by definition of $\bigwedge_{I',X}$ that $\bigwedge_{I',X}((f \times 1_X)^*(A)) = I'$. In particular $* \in \bigwedge_{I',X}((f \times 1_X)^*(A))$ such that by the naturality condition $* \in f^*(\bigwedge_{I,X}(A))$ and therefor $i = f(*) \in \bigwedge_{I,X}(A)$. This proves the inverse inclusion.

We now consider the case where $I = \mathbb{R}_{\geq 0}$ equipped with the regular Euclidean topology and $X = \mathbb{R}_{>0}$ equipped with the discrete topology. We consider the subset $A = \{(x, y) \in I \times X | x < y\}$. Note that we may also write this set as

$$A = \bigcup_{y \in \mathbb{R}_{>0}} [0, y) \times \{y\} = \bigcup_{y \in \mathbb{R}_{>0}} ((-y, y) \cap \mathbb{R}_{\geq 0}) \times \{y\} = \bigcup_{y \in X} ((-y, y) \cap I) \times \{y\}$$

from which it is clear that A is open in $I \times X$. We now show $\bigwedge_{I,X}(A) = \{0\}$. By the preceding we have $\bigwedge_{I,X}(A) = \{j \in I | \forall x \in X((i,x) \in A)\}$. By our choice for X, A we now get $0 \in \bigwedge_{I,X}(A)$. Let now $x \in I$ be not equal to 0. Then $x > 0$ and consequently $x > \frac{x}{2} > 0$. As $\frac{x}{2} \in \mathbb{R}_{>0} = I$, but not $x < \frac{x}{2}$ so $(x, \frac{x}{2}) \notin A$ we have by our expression for $\bigwedge_{I,X}(A)$ that $x \notin \bigwedge_{I,X}(A)$. We conclude that 0 is the unique element of $\bigwedge_{I,X}(A)$, or equivalently $\bigwedge_{I,X}(A) = \{0\}$. By $\mathbb{R}_{\geq 0}$ being haussdorff we have that all singleton and therefor in particular $\{0\}$ is closed. Also, $\mathbb{R}_{\geq 0}$ is path connected and therefor also connected such that the only clopen subsets are $\emptyset, \mathbb{R}_{\geq 0}$. In particular $\{0\}$ is not open. However we defined $\bigwedge_{I,X}(A) = \{0\}$ as an interior so this is a contradiction. We conclude that Top does not have universal quantifiers. $\triangle$

Exercise 2. (7 points) In this exercise we consider the category Grp of groups and group morphisms. We make it into a prop-category by taking for X a group, $\text{Prop}_{\text{Grp}}(X)$ to be the set of subgroups ordered by inclusion and for $f : Y \to X$ a group morphism, the pullback operation $f^* : \text{Prop}_{\text{Grp}}(X) \to \text{Prop}_{\text{Grp}}(Y)$ to be defined by taking the preimage through f. Show that Grp has equality.

Solution. We first show that Grp has finite meets. Let X be a group and let U, V be subgroups. Then $U \cap V$ is again a subgroup. Furthermore if W is another subgroup then considering U, V, W as set we see

$$W \subseteq U, W \subseteq V \Leftrightarrow W \subseteq U \cap V$$

such that $U \cap V$ is the meet of U, V. Let now U again be a subgroup of X. Then $U \subseteq X$. We see that X is a maximal element of $\text{prop}_{\text{Grp}}(X)$. For $f : Y \to X$ a group homomorphism we have $f^*(X) = Y$ and $f^*(U \cap V) = f^*(U) \cap f^*(V)$ such that the maximal element and binary meets are preserved. We conclude that X has finite meets.

Let again X be a group. We define a subset $\text{Eq}_X = \{(x, x) | x \in X\}$. It is easily seen that this gives a subgroup of $X \times X$.

Let now I, X be groups and let A be a subgroup of $I \times X$ and B a subgroup of $I \times X \times X$. Suppose $A \subseteq (1_I \times \Delta_X)^*(B)$. We need to show $(1_I \times \pi_1)^*(A) \cap \pi_2^*(\text{Eq}_X) \subseteq B$. Let $(i, x, y) \in (1_I \times \pi_1)^*(A) \cap \pi_2^*(\text{Eq}_X)$. Then $(i, x, y) \in (1_I \times \pi_1)^*(A)$ and $(i, x, y) \in \pi_2^*(\text{Eq}_X)$. Consequently $(i, x) \in A \subseteq (1_I \times \Delta_X)^*(B)$ and $(x, y) \in \text{Eq}_X$. We now find $(i, x, x) \in B$ and $x = y$. Combining this gives $(i, x, y) \in B$. This proves $(1_I \times \pi_1)^*(A) \cap \pi_2^*(\text{Eq}_X) \subseteq B$.

Suppose conversely that $(1_I \times \pi_1)^*(A) \cap \pi_2^*(\text{Eq}_X) \subseteq B$. We need to show $A \subseteq (1_I \times \Delta_X)^*(B)$. Let

$(i,x) \in A$. Note that $(1_I \times \pi_1)(i,x,x) = (i,x) \in A$ and $\pi_2(i,x,x) = (x,x) \in \mathrm{Eq}_X$. Consequently

$$(i,x,x) \in (1_I \times \pi_1)^*(A) \cap \pi_2^*(\mathrm{Eq}) \subseteq B.$$

We now find that

$$(1_I \times \Delta_X)(i,x) = (i,x,x) \in B$$

and therefor $(i,x) \in (1_I \times \Delta_X)^*(B)$. This proves $A \subseteq (1_I \times \Delta_X)^*(B)$. We conclude that

$$A \subseteq (1_I \times \Delta_X)^*(B) \Leftrightarrow (1_I \times \pi_1)^*(A) \cap \pi_2^*(\mathrm{Eq}_X) \subseteq B$$

which shows that Grp has equality. △

Exercise 3. (3 points) Prove the statement preceding Proposition 5.7.1 that the assignment $(-)^*$ is functorial.

Solution. Let

$$\Gamma'' \xrightarrow{\delta=[N_1,\ldots,N_n]} \Gamma' = [x_1' : \sigma_1', \ldots, x_n' : \sigma_n'] \xrightarrow{\gamma=[M_1,\ldots,M_m]} \Gamma = [x_1 : \sigma_1, \ldots, x_m : \sigma_m]$$

be morphisms in $\mathcal{C}\ell(Th)$. Let $[\phi] \in \mathrm{prop}_{\mathcal{C}\ell(Th)}(\Gamma)$. Then we find

$$1_\Gamma^*([\phi]) = [x_1, \ldots, x_m]^*([\phi]) = [\phi[x_1/x_1, \ldots, x_n/x_n]] = [\phi] = 1_{\mathrm{prop}_{\mathcal{C}\ell(Th)}}([\phi])$$

such that in general $1_\Gamma^* = 1_{\mathrm{prop}_{\mathcal{C}\ell(Th)}}([\phi])$ and

$$\begin{aligned}
(\delta^* \circ \gamma^*)([\phi]) &= \delta^*([\phi[M_1/x_1, \ldots, M_n/x_n]]) \\
&= [\phi[M_1/x_1, \ldots, M_n/x_n][N_1/x_1', \ldots, N_m, x_m']]) \\
&= [\phi[M_1[N_1/x_1', \ldots, N_n/x_n'], \ldots, M_m[N_1/x_1', \ldots, N_n/x_n']]] \\
&= [M_1[N_1/x_1', \ldots, N_n/x_n'], \ldots, M_m[N_1/x_1', \ldots, N_n/x_n']]^*([\phi]) \\
&= (\gamma \circ \delta)^*([\phi])
\end{aligned}$$

such that in general $(\gamma \circ \delta)^* = \delta^* \circ \gamma^*$. △