Homework 7

March 24, 2024

Exercise 1. (3 points)

Let $\mathcal{C}$ be a small category with pullbacks. Grothendieck topologies are supposed to represent choices of 'covering families'. This suggests the following construction. Assign to each $C \in \mathcal{C}$ the following collection of sieves $J(C) = \{S \subset y(C) \mid S \text{ jointly epi}\}$.

For a family of arrows A (all with the codomain C) we write $\mathrm{Pb}_f(A) = \{f^*g \mid g \in A\}$ for some $f : D \to C$, where f^*g is given by the pullback square

$$\begin{array}{ccc} \bullet & \xleftarrow{f^*g} & \bullet \\ {\scriptstyle g}\big\downarrow & \lrcorner & \big\downarrow \\ C & \xleftarrow[f]{} & D \end{array}$$

Prove that if for any jointly epi family A the set $\mathrm{Pb}_f(A)$ is jointly epi then J is a Grothendieck topology on $\mathcal{C}$.

Exercise 2. (7 points)

Prove the uniqueness part of the universal property of coproducts of sheaves. In other words, prove that for sheaves $\mathcal{F}_i$ the morphisms $\sigma_i : \mathcal{F}_i \to \sum_i \mathcal{F}_i$ are jointly epi.

Exercise 3. (7 points)

Prove the case for $\vee$ of the lemma on page 11. Is the induction hypothesis necessary for this case?