Seminar on Models of Intuitionism

Hand-in exercise 4

March 9, 2017 (due March 16)

In these exercises, T denotes the Baire space.

Exercise 1. We interpret the predicate '$x \in \mathbb{Q}$' on the toplogical model by setting

$$[\![\xi \in \mathbb{Q}]\!] = \bigcup_{q \in \mathbb{Q}} [\![\xi = q]\!],$$

for $\xi \in \mathcal{R}$. Inside the brackets, q denotes the constant function $T \to \mathbb{R}$ taking the value q.

(a) *1 point.* Let $A(x)$ be a formula in one free variable (possibly with parameters in $\mathcal{R}$) and suppose that $[\![\forall xy\,(A(x) \wedge x = y \to A(y))]\!] = T$. Show that

$$[\![\exists x\,(x \in \mathbb{Q} \wedge A(x))]\!] = \bigcup_{q \in \mathbb{Q}} [\![A(q)]\!].$$

(b) *2 points.* Show that the sentence

$$\forall x, y(x < y \to \exists z\,(z \in \mathbb{Q} \wedge (x < z < y))$$

is valid in the toplogical model.

(c) *2 points.* Give an example of a continuous function $\xi : T \to \mathbb{R}$ such that

$$[\![\xi \in \mathbb{Q}]\!] \neq \mathrm{Int}\{t \in T \colon \xi(t) \in \mathbb{Q}\}.$$

Exercise 2. Let $\xi \in \mathcal{R}$ and $\varphi \in \mathcal{R}^{\mathcal{R}}$ and consider the continuous function $\alpha : T \times (0, \infty) \to \mathbb{R}$ given by

$$\alpha(s, \delta) = \sup\{|\Phi(s, \xi(s)) - \Phi(s, a)| \colon a \in \mathbb{R} \text{ and } |\xi(s) - a| \leq \delta\}.$$

Here $\Phi : T \times \mathbb{R} \to \mathbb{R}$ is the continuous function associated to φ, as defined in the lecture.

(a) *1 point.* Explain why $\alpha(s, \delta)$ is always defined and why α is continuous, and show that for a fixed $s \in T$, we have $\alpha(s, \delta) \to 0$ when $\delta \to 0$.

Let $\varepsilon \in \mathcal{R}$. Suppose we have a $t \in T$ such that $\varepsilon(t) > 0$. By exercise (a), there is a $\delta > 0$ such that $\alpha(t, \delta) < \varepsilon(t)$.

(b) *2 points.* Show that for such a δ, we have:

$$t \in \mathrm{Int} \bigcap_{\eta \in \mathcal{R}} \mathrm{Int}(\{s \in T \colon |\xi(s) - \eta(s)| \geq \delta\} \cup \{s \in T \colon |\Phi(s, \xi(s)) - \Phi(s, \eta(s))| < \varepsilon(s)\}).$$

(c) *2 points.* Show that the sentence

$$\forall f \forall x \forall \varepsilon\,(\varepsilon > 0 \to \exists \delta\,(\delta > 0 \wedge \forall y\,(x - \delta < y < x + \delta \to f(x) - \varepsilon < f(y) < f(x) + \varepsilon)))$$

is valid in the topological model. Here the addition symbol is interpreted in the topological model as pointwise addition of continuous functions $T \to \mathbb{R}$, and similarly for substraction.