Exercise 1

Define α as follows:

$$\alpha(e) = \mu n \leq j_2(e)(\forall_{x \leq j_2(e)}(x \neq n \to \exists_{y \leq f(e)} T(j_1(e), x, y))),$$

with f defined as

$$f(e) = \mu n(\exists_{x \leq j_2(e)} \forall_{y \leq j_2(e)}(x \neq y \to \exists_{z \leq n} T(j_1(e), y, z))).$$

To see that α is partial recursive, note that we can rewrite the above equations as

$$\alpha(e) = \mu n \leq j_2(e)(\forall_{x \leq j_2(e)}(x = n \vee \exists_{y \leq f(e)} T(j_1(e), x, y))),$$

and

$$f(e) = \mu n(\exists_{x \leq j_2(e)} \forall_{y \leq j_2(e)}(x = y \vee \exists_{z \leq n} T(j_1(e), y, z))).$$

respectively. We see that α (and necessarily also f) is constructed by means of minimalisation, bounded quantification and disjunction of partial recursive predicates. Hence, we may conclude α itself is partial recursive as well.

To see that α meets the requirements, let e be a natural number such that V_e contains only a single element k. That is, e is a number such that $\varphi_{j_1(e)}$ is undefined on only a single number k smaller or equal to $j_2(e)$. Then certainly $f(e)$ is defined, since we can simply take it to be the least upper bound of $\{z : T(j_1(e), y, z) \wedge y \leq j_2(e) \wedge k \neq y\}$. Consequently, $\alpha(e)$ will also be defined and, in particular, will be equal to the number k.

Grading:
1 point for giving an appropriate α.
1 point for showing this α meets the requirements.

Exercise 2

a) Let $A(x) \equiv \exists y Txxy$ and suppose we can derive $\forall x(\neg\exists y Txxy \vee \neg\neg\exists y Txxy)$. Applying our knowledge of realizability, we see that the preceding assumption means that $\forall x(\neg\exists y Txxy \vee \neg\neg\exists y Txxy)$ is realizable in Kleene's sense. That is, there exists a number n such that

$$n \text{ realizes } \forall x(\neg\exists y Txxy \vee \neg\neg\exists y Txxy),$$

which means

$$\text{for all } m : \varphi_n(m) \text{ realizes } \neg\exists y Txxy \vee \neg\neg\exists y Txxy \text{ and } \varphi_n(m)\downarrow$$

i.e.

$$\text{for all } m : j_1(\varphi_n(m)) = 0 \text{ implies } j_2(\varphi_n(m)) \text{ realizes } \neg\exists y Txxy \text{ and}$$
$$j_1(\varphi_n(m)) \neq 0 \text{ implies } j_2(\varphi_n(m)) \text{ realizes } \neg\neg\exists y Txxy \text{ and } \varphi_n(m)\downarrow$$

The first implication tells us that if $j_1(\varphi_n(m)) = 0$ then there is no realizer for $\exists y Txxy$, i.e. $\varphi_x(x)$ is undefined. Similarly, the second implication tells us that if $j_1(\varphi_n(m)) \neq 0$ then there is no realizer for $\neg\exists y Txxy$. From the latter fact, we can infer that there must exist some y such that $Txxy$, i.e. $\varphi_x(x)$ is defined. This, however, implies that the function $j_1 \circ \varphi_n$ decides the diagonal halting set and we have arrived at a contradiction.

Grading:
1 point for linking derivability to realizability.
0.5 points for selecting the right formula A.
1 point for deriving the contradiction.

b) Suppose there exists a recursive set C such that $B \subseteq C$ and $A \subseteq \mathbb{N} \backslash C$. Since C is recursive, there exists an index i such that φ_i is the characteristic function of C. Next, note that if $x \in A$, then $x \notin C$ and thus $\varphi_i(x) = 1$. Similarly, if $x \in B$ then $x \in C$ and hence $\varphi_i(x) = 1$. Now, suppose $i \in C$. Then $\varphi_i(i) = 0$ and thus, by definition of A, we have $i \in A$, which implies $i \notin C$: a contradiction. In the same vein, we arrive at a contradiction in case $i \notin C$. We conclude A and B are recursively inseparable.

Grading:
0.5 points for showing $x \in A, x \in B$ imply $\varphi_i(x) = 1, \varphi_i(x) = 0$ respectively.
1 point for considering the index i of the characteristic function of C.
1 point for showing $i \in C$ and $i \notin C$ both lead to a contradiction.

c) Let $\alpha(x,y), \beta(x,y)$ be the characteristic functions of the sets $\{(x,y) : Txxy \wedge U(y) = 0\}$ and $\{(x,y) : Txxy \wedge U(y) = 1\}$ respectively. Then the sets $\{x : \exists y \alpha(x,y)\}$ and $\{x : \exists y \beta(x,y)\}$ are identical to the sets A and B from exercise 2b respectively and, hence, are recursively inseparable. Now, suppose

$$\forall x(\neg(\exists y(\alpha(x,y) = 0) \wedge \exists y(\beta(x,y) = 0)) \to \neg\exists y(\alpha(x,y) = 0) \vee \neg\exists y(\beta(x,y) = 0)) \tag{1}$$

is derivable in **HA** + CT_0. Because A and B are recursively inseparable, we have

$$\forall x(\neg(\exists y(\alpha(x,y) = 0) \wedge \exists y(\beta(x,y) = 0)). \tag{2}$$

From (1) and (2) it follows that

$$\forall x(\neg\exists y(\alpha(x,y) = 0) \vee \neg\exists y(\beta(x,y) = 0)). \tag{3}$$

is derivable, which implies it is also realizable in Kleene's sense. That is, there exists a number n such that

$$n \text{ realizes } \forall x(\neg\exists y(\alpha(x,y)=0) \vee \neg\exists y(\beta(x,y)=0)).$$

Applying the definition of realizability, we get

$$\text{for all } m : \varphi_n(m) \text{ realizes } (\neg\exists y(\alpha(x,y)=0) \vee \neg\exists y(\beta(x,y)=0)) \text{ and } \varphi_n(m)\downarrow$$

$$\begin{aligned}\text{for all } m : & j_1(\varphi_n(m)) = 0 \text{ implies } j_2(\varphi_n(m)) \text{ realizes } \neg\exists y(\alpha(x,y)=0) \text{ and} \\ & j_1(\varphi_n(m)) \neq 0 \text{ implies } j_2(\varphi_n(m)) \text{ realizes } \neg\exists y(\beta(x,y)=0) \text{ and } \varphi_n(m)\downarrow\end{aligned}$$

Hence, we have a recursive function φ_n such that $\varphi(x) = 0$ implies that there exists no y such that $\alpha(x,y) = 0$ and $\varphi_n(x) \neq 0$ implies that there exists no y such that $\beta(x,y) = 0$. That is, if $\exists y(\alpha(x,y) = 0)$ then $\varphi_n(x) \neq 0$ and if $\exists y(\beta(x,y) = 0)$ then $\varphi_n(x) = 0$.

Now, let C be the set with characteristic function φ_n. Clearly, C is recursive. Moreover, if $x \in A$ then, by definition of A, $\exists y(\alpha(x,y) = 0)$. Hence, $\varphi_n(x) \neq 0$, which means $x \in \mathbb{N}\backslash C$. Thus,we see $A \subseteq \mathbb{N}\backslash C$. Alternatively, if $x \in B$ then $\exists y(\beta(x,y) = 0)$ and thus $\varphi_n(x) = 0$. We infer that $x \in C$ and hence $B \subseteq C$. This, however, contradicts the fact that A and B are recursively inseparable. We conclude (1) is not derivable in **HA** + CT_0.

Grading:
1 point for finding appropriate functions α and β.
1 point for showing the existence of φ_n.
1 point for showing the recursive set C separates A and B.