Seminar on Models of Intuitionism

Hand-in exercise 11: model solution

11 May

Exercise 1.

(a) Let X and Y G-sets. The G-actions on X and Y induce a G-action on the cartesian product of X and Y. Namely, for $(x, y) \in X \times Y$, put $g \cdot (x, y) = (g \cdot x, g \cdot y)$. One readily verifies that this indeed satisfies the axioms for a group action. So $X \times Y$ is a G-set.

We show that the projection $\pi\colon X \times Y \to X$ is a G-map. Let $g \in G$ and $(x, y) \in X \times Y$. Then $g \cdot \pi(x, y) = g \cdot x = \pi(g \cdot x, g \cdot y) = \pi(g \cdot (x, y))$. So π is indeed a G-map. Similarly, the projection $\pi'\colon X \times Y \to Y$ is also a G-map.

Now suppose we have a G-set Z and G-maps $a\colon Z \to X$ and $b\colon Z \to Y$. We claim that $\langle a, b\rangle\colon Z \to X \times Y$ given by $z \mapsto (a(z), b(z))$ is a G-map. To this end, let $g \in G$ and $z \in Z$. Note that $g \cdot \langle a, b\rangle(z) = g \cdot (a(z), b(z)) = (g \cdot a(z), g \cdot b(z)) = (a(g \cdot z), b(g \cdot z)) = \langle a, b\rangle(g \cdot z)$, since a and b are G-maps. Hence, $\langle a, b\rangle$ is indeed a G-map.

Also, we immediately see that $\pi \circ \langle a, b\rangle = a$ and $\pi' \circ \langle a, b\rangle = b$. Finally, that $\langle a, b\rangle$ is the unique map with these properties follows immediately from the corresponding fact in Set. ∎

Each paragraph is worth half a point.

(b) Suppose X and Y are G-sets. We need to define a G-action on Y^X such that the map $\varepsilon\colon X \times Y^X \to Y$ given by $(x, f) \mapsto f(x)$ is a G-map. That is: $g \cdot \varepsilon(x, f) = g \cdot f(x)$ should equal $\varepsilon(g \cdot (x, f)) = \varepsilon((g \cdot x, g \cdot f)) = (g \cdot f)(g \cdot x)$ for any $x \in X$ and $f \in Y^X$. This suggests putting $g \cdot f = (x \mapsto g \cdot f(g^{-1} \cdot x))$. For then, given any $x \in X$, we have $(g \cdot f)(g \cdot x) = g \cdot f(g^{-1} \cdot (g \cdot x)) = g \cdot f((g^{-1}g) \cdot x) = g \cdot f(e \cdot x) = g \cdot f(x)$, so ε is a G-map.

We verify that this is indeed a group action on Y^X. Let $f \in Y^X$ and $x \in X$. First of all, $(e \cdot f)(x) = e \cdot f(e^{-1} \cdot x) = f(x)$, so $e \cdot f = f$. Furthermore, for all $g, h \in G$, we have

$$\begin{aligned}(h \cdot (g \cdot f))(x) &= h \cdot ((g \cdot f)(h^{-1} \cdot x)) = h \cdot (g \cdot f(g^{-1} \cdot (h^{-1} \cdot x))) = (hg) \cdot f((g^{-1}h^{-1}) \cdot x) \\ &= (hg) \cdot f((hg)^{-1} \cdot x) = ((hg) \cdot f)(x),\end{aligned}$$

so $h \cdot (g \cdot f) = (hg) \cdot f$, as desired.

We proceed by checking the universal property. Let Z be a G-set and $f\colon X \times Z \to Y$ a G-map. We claim that $\tilde{f}\colon Z \to Y^X$ given by $z \mapsto (x \mapsto f(x, z))$ is a G-map. Let $g \in G$ and $z \in Z$ be arbitrary. Note that since f is a G-map, we have for all $x \in X$ that:

$$\left(g \cdot \tilde{f}(z)\right)(x) = g \cdot f(g^{-1} \cdot x, z) = f(g \cdot (g^{-1} \cdot x), g \cdot z) = f(e \cdot x, g \cdot z) = f(x, g \cdot z) = \tilde{f}(g \cdot z)(x),$$

so $g \cdot \tilde{f}(z) = \tilde{f}(g \cdot z)$. Thus, $\tilde{f}$ is indeed a G-map.

Finally, we know that $\varepsilon \circ \langle \pi, \tilde{f} \circ \pi'\rangle = f$ as we have seen this in Set. That $\tilde{f}$ is the unique arrow with this property follows from the corresponding fact in Set. ∎

One point each for the first two paragraphs. Half a point for each of the final two paragraphs.

Exercise 2.

(4) Let $A \overset{f}{\Rightarrow} \top$ be a deduction. The equation $f = !_A$ identifies $A \overset{f}{\Rightarrow} \top$ with $\dfrac{}{A \Rightarrow \top}\,\textsc{true}$. In other words, there is, up to equivalence, exactly one deduction of the entailment $A \Rightarrow \top$, namely the one given by the TRUE-rule. ■

(5) Let $C \overset{f}{\Rightarrow} A$ and $C \overset{g}{\Rightarrow} B$ be deductions. The equation $\pi \circ \langle f, g \rangle = f$ identifies

$$\dfrac{\dfrac{C \overset{f}{\Rightarrow} A \qquad C \overset{g}{\Rightarrow} B}{C \Rightarrow A \wedge B}\,\wedge\text{R} \qquad \dfrac{}{A \wedge B \Rightarrow A}\,\wedge\text{L1}}{C \Rightarrow A}\,\text{CUT}$$

with $C \overset{f}{\Rightarrow} A$ itself. ■

(7) Let $C \overset{h}{\Rightarrow} A \wedge B$ be a deduction. The equation $\langle \pi \circ h, \pi' \circ h \rangle = h$ identifies

$$\dfrac{\dfrac{C \overset{h}{\Rightarrow} A \wedge B \qquad \dfrac{}{A \wedge B \Rightarrow A}\,\wedge\text{L1}}{C \Rightarrow A}\,\text{CUT} \qquad \dfrac{C \overset{h}{\Rightarrow} A \wedge B \qquad \dfrac{}{A \wedge B \Rightarrow B}\,\wedge\text{L2}}{C \Rightarrow B}\,\text{CUT}}{C \Rightarrow A \wedge B}\,\wedge\text{R}$$

with $C \overset{h}{\Rightarrow} A \wedge B$ itself. ■

(10) Let $A \vee B \overset{h}{\Rightarrow} C$ be a deduction. The equation $[h \circ \kappa, h \circ \kappa'] = h$ identifies

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A \vee B}\,\vee\text{R1} \qquad A \vee B \overset{h}{\Rightarrow} C}{A \Rightarrow C}\,\text{CUT} \qquad \dfrac{\dfrac{}{B \Rightarrow A \vee B}\,\vee\text{R2} \qquad A \vee B \overset{h}{\Rightarrow} C}{B \Rightarrow C}\,\text{CUT}}{A \vee B \Rightarrow C}\,\vee\text{L}$$

with $A \vee B \overset{h}{\Rightarrow} C$ itself. ■

(12) Let $C \overset{k}{\Rightarrow} A \to B$ be a deduction. The equation $(\varepsilon \circ \langle \pi, k \circ \pi' \rangle)^{\sim} = k$ identifies

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge C \Rightarrow A}\,\wedge\text{L1} \qquad \dfrac{\dfrac{}{A \wedge C \Rightarrow C}\,\wedge\text{L2} \qquad C \overset{k}{\Rightarrow} A \to B}{A \wedge C \Rightarrow A \to B}\,\text{CUT}}{A \wedge C \Rightarrow A \wedge (A \to B)}\,\wedge\text{R} \qquad \dfrac{}{A \wedge (A \to B) \Rightarrow B}\,\to\text{L}}{A \wedge C \Rightarrow B}\,\text{CUT}}{C \Rightarrow A \to B}\,\to\text{R}$$

with $C \overset{k}{\Rightarrow} A \to B$ itself. ■