Seminar on models of Intuitionism

Model solution 12

May 24, 2017

Solution to exercise 1.

(a) Consider the poset S of all filters on H which contain F but do not contain x, where these filters are ordered by inclusion. Let I be some index set and $\{C_i \mid i \in I\}$ a chain in S. Since $F \in S$, the empty chain has an upper bound in S. If I is not empty, we consider $C = \bigcup_{i \in I} C_i$. Notice that $1 \in C$ but $0 \notin C$. Also, if $y \in C$ then $y \in C_i$ for some i. Hence if $z \geq y$, then $z \in C_i$ because C_i is a filter, and hence $z \in C$. So C is upwards closed. Now suppose $y, z \in C$, then $y \in C_i$ and $z \in C_j$ for some $i, j \in I$, and since $C_i \subseteq C_j$ or $C_j \subseteq C_i$, we can say without loss of generality that $x, y \in C_i$, and hence $x \wedge y \in C_i$, so $x \wedge y \in C$. So C is closed under meets, and hence C is indeed a filter. Since $C \subset H$, we can conclude that any chain in S has an upper bound in H, and hence by Zorns lemma S has a maximal element P. We will now prove that P is a prime filter. So suppose $a, b \in H$ and $a \vee b \in P$. Suppose that there are elements $p, q \in P$ such that $p \wedge a \leq x$ and $q \wedge b \leq x$. Then we notice that $(p \wedge q) \wedge (a \vee b) \leq x$. But since $p \wedge q \in P$ and $a \vee b \in P$, we would find $x \in P$, which is impossible. Hence we either have $\forall p \in P(p \wedge a \nleq x)$ or $\forall p \in P(p \wedge b \nleq x)$. Say without loss of generality that we are in the first case. Then consider the filter generated by P and a. This filter contains F but it does not contain x, and hence it must be P itself, since P is contained in it, and P is maximal with respect to this property. And therefore we find that $a \in P$. So P is indeed prime, concluding the proof of the prime filter existence theorem. $\square$
Correctly using Zorn's lemma was worth 1 point, and proving that a maximal filter is a prime filter was worth 1 point. $\frac{1}{4}$ points were deduced for small mistakes such as forgetting the empty chain.

(b) First we will prove that if P is a prime filter and $i : H \to 2$ is defined by $i(x) = 1$ iff $x \in P$, then $i \in [H, 2]$. Notice that $i(1) = 1$, and if $x \leq y$ then if $x \in P$ we also have $y \in P$, hence $i(x) \leq i(y)$. From this it quickly follows that for all $x, y \in H$, we have $i(x \wedge y) \leq i(x) \wedge i(y)$. Now suppose $i(x) = i(y) = 1$, then $x, y \in P$ hence $x \wedge y \in P$, so $i(x \wedge y) = 1$. Hence $i(x) \wedge i(y) \leq i(x \wedge y)$, so i preserves meets. Since i is order-preserving, we also see that $i(x) \vee i(y) \leq i(x \vee y)$. Now suppose $i(x \vee y) = 1$, then $x \vee y \in P$, and since P is prime we find that $x \in P$ or $y \in P$, hence $i(x) \vee i(y) = 1$. So i also preserves joins. We can conclude that i is indeed an almost lattice homomorphism.
Now let $x, y \in H$ be such that $x \neq y$. Then there is a filter F containing one of these elements, but not the other (the upwards closure of the greatest of the two, or of one of the two if we cannot compare them). So say without loss of generality that $x \in F$ and $y \notin F$. Then by the prime filter existence theorem there is a prime filter P such that $F \subseteq P$ and $y \notin P$. Now let $i \in [H, 2]$ be defined by $i(z) = 1$ iff $z \in P$. Then we notice that $i(x) \neq i(y)$, and hence $e(x)(i) \neq e(y)(i)$. So e is indeed injective. $\square$
Proving that a prime filter induces an almost lattice homomorphism was worth 1.5 points. Giving the right filter (the upwards closure of x or y) and explaining why it

contains just one of these two elements was worth $\frac{1}{2}$ point. The rest was worth 1 point. Again $\frac{1}{4}$ points were deduced for small mistakes.

Solution to exercise 2. We first note that since $F(\mathbf{t}) = \{*\}$ there is an arrow $F(\mathbf{t}) \to X$ for a $\mathbb{Z}$-set X if and only if there is a fixed point in X. That is, there is some $x \in X$ such that $n \cdot = x$ for all $n \in \mathbb{Z}$. This is easily seen by considering the arrow $f : \{*\} \to X$ given by $f(*) = x$.
For minor mistakes $\frac{1}{4}$ point is subtracted. For example: forgetting to mention why a certain element is a fixed point.

(a) We will show that $F(\neg A \vee \neg\neg A)$ always contains a fixed point. Note that

$$F(\neg A \vee \neg\neg A) = F(\neg A) + F(\neg\neg A) = \emptyset^{F(A)} + \emptyset^{(\emptyset^{F(A)})}.$$

Suppose that $F(A) = \emptyset$, then $\emptyset^{F(A)}$ is a singleton. Otherwise we have $\emptyset^{F(A)} = \emptyset$, hence $\emptyset^{(\emptyset^{F(A)})}$ is a singleton. Since the singleton $\mathbb{Z}$-set contains exactly one element, that element has to be a fixed point. So we either have a fixed point in $\emptyset^{F(A)}$ or a fixed point in $\emptyset^{(\emptyset^{F(A)})}$. In both cases we find a fixed point in $\emptyset^{F(A)} + \emptyset^{(\emptyset^{F(A)})}$.

(b) Consider $\mathbb{Z}/n\mathbb{Z}$ for any $n > 1$ (e.g. $n = 2$), where the group action is addition modulo n. Then for a functor with $F(A) = \mathbb{Z}/n\mathbb{Z}$ we have that

$$F(A \vee \neg A) = F(A) + \emptyset^{F(A)}.$$

Since $\emptyset^{F(A)} = \emptyset$, this is isomorphic to $F(A)$. Since $F(A)$ has no fixed point we cannot have an arrow $F(\mathbf{t}) \to F(A)$ and thus no arrow $F(\mathbf{t}) \to F(A \vee \neg A)$.