Exercise 1

a) *(4 points)* *Solution.* In what follows, let $\forall k_{max} \in k$ be a shorthand for "for every maximal node succeeding k". Let $\mathscr{B}$ be a finite Beth model over a relational, first-order language L. Following the hint, we first prove for a given node k in $\mathscr{B}$ and a given $L(D(k))$-sentence A, that if $\forall k_{max} \in k$: $k_{max} \Vdash A$, then $k \Vdash A$. We proceed by induction, limiting ourselves to the initial step and the induction step for $\vee$.

Initial step: Let A be an atomic $L(D(k))$-sentence and suppose that $\forall k_{max} \in k$: $k_{max} \Vdash A$. By definition, we know that $k \Vdash A$ iff $\forall k_{max} \in k$, k_{max} has an initial segment k' such that $k' \Vdash A$. Now, it follow by assumption that $\forall k_{max} \in k$ such an initial segment exists, viz. the node k_{max} itself. Hence, $k \Vdash A$. *(1 point)*

Induction step, $\vee$-case: Let A be of the form $B \vee C$ and suppose $\forall k_{max} \in k$: $k_{max} \Vdash B \vee C$. Since the nodes k_{max} are maximal, the latter assumption immediately implies that

$$\forall k_{max} \in k : k_{max} \Vdash B \text{ or } k_{max} \Vdash C. \qquad (\star)$$

Now, by definition, $k \Vdash B \vee C$ iff $\forall k_{max} \in k$: k_{max} has an initial segment k' such that either $k' \Vdash B$ or $k' \Vdash C$. Looking at $(\star)$, we see that $\forall k_{max} \in k$ such an initial segment exists, viz. the node k_{max} itself. Hence, $k \Vdash B \vee C$. Note that we do not require the induction hypothesis for this step. *(1 point)*

Turning now to the desired proof, let A be a CQC-theorem in the language L. Using the equivalence (1) as stated in the hint, we then know that $\Gamma \vdash_{IQC} A$, with

$$\Gamma = \{\neg\neg B \to B \mid B \text{ is an } L\text{-sentence}\}.$$

By soundness, the latter fact implies that $\Gamma \Vdash A$. Thus, if we can show that $\mathscr{B} \Vdash \Gamma$, we can conclude that $\mathscr{B} \Vdash A$ and we are done. *(1 point)*

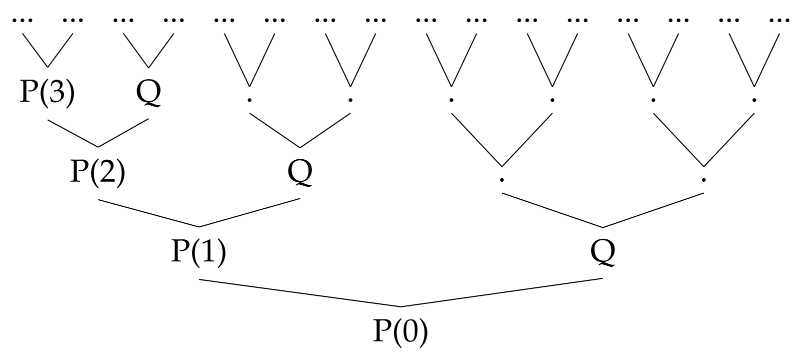
To see that indeed $\mathscr{B} \Vdash \Gamma$, let $\neg\neg B \to B$ be an arbitrary sentence in Γ and k an arbitrary node in $\mathscr{B}$. Now, let k_{max} be an arbitrary maximal node succeeding k. Since k_{max} is maximal, every universal quantifier in the truth definition for $\neg\neg B \to B$ will vanish, yielding

$$k_{max} \Vdash \neg\neg B \to B \text{ iff (not: not: } k_{max} \Vdash B) \text{ implies } k_{max} \Vdash B,$$

which is trivially true. So, the sentence $\neg\neg B \to B$ is true at every maximal node succeeding k and, hence, it is true at k. We conclude that $\mathscr{B} \Vdash \Gamma$. *(1 point)*

The obtained result may be contrasted with your answers to question 2 of the first homework set. We see that, in general, we have no trouble refuting classical theorems using finitely large Kripke models.

b) *(2 points) Solution.* Let us consider the simple case where $P(x)$ is an atomic formula and Q an atomic sentence. Consider the Beth model having constant domain $\mathbb{N}$, with the underlying spread and forcing relation given by: *(1½ points)*



Here, each node is labeled by each *new* atomic sentence holding true at that node. *(½ points)* The presence of the left-most branch ensures the consequent of (2) will be false.

Exercise 2. In this exercise you are asked to verify the details of the proof of Theorem 6.13, which states that any Ha can be embedded in a cHa preserving $\wedge, \vee, \to, \bot$ and all existing meets and joins.

(a) (*1 point*) Let $(A, \leq)$ be a complete lattice satisfying the infinitary distributive law

$$D \qquad \forall a \in A\ \forall B \subseteq A, \quad a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\}.$$

Show that $(A, \leq)$ is a cHa.

Solution. We have to show that for each $a, b \in A$ the set $B = \{x \mid x \wedge a \leq b\}$ has a greatest element. By D, we have

$$\bigvee B \wedge a = \bigvee \{x \wedge a \mid x \wedge a \leq b\} \leq b.$$

This means that $\bigvee B \in B$ and thus $\bigvee B$ is the greatest element of B. □

Let $\Theta = (A, \leq)$ be a Ha. A *c-ideal* of Θ is a subset $I \subseteq A$ satisfying

$\bot \in I$;

If $b \in I$ and $a \leq b$, then $a \in I$;

If $X \subseteq I$ and $\bigvee X$ exists in Θ, then $\bigvee X \in I$.

Now let Ω be the poset of c-ideals of Θ ordered by inclusion.

(b) (*1 point*) Show that Ω is a complete lattice.

Solution. For Ω to be a complete lattice it must be the case that each subset $\{I_b\}_{b \in B}$ of Ω has a meet and a join.

The meet of $\{I_b\}_{b \in B}$ is the greatest c-ideal contained in all I_b. We will show that their intersection is itself a c-ideal and thus that $\bigwedge_{b \in B} I_b = \bigcap_{b \in B} I_b$. First notice that $\bot \in I_b$ for a $b \in B$ and thus $\bot \in \bigcap_{b \in B} I_b$. Now let $x \in \bigcap_{b \in B} I_b$ and $y \in \Theta$ such that $y \leq x$. We have for each $b \in b$ that $x \in I_b$ and thus, by the fact that I_b is a c-ideal, also $y \in I_b$. It follows that $y \in \bigcap_{b \in B} I_b$, as required. Finally, suppose that $X \subseteq \bigcap_{b \in B} I_b$ and $\bigvee X$ exists in Θ. Again, since each I_b is a c-ideal containing X, we have $\bigvee X \in I_b$ for each b. We find $\bigvee X \in \bigcap_{b \in B} I_b$.

Similarly, the join of $\{I_b\}_{b \in B}$ is the least c-ideal containing all I_b. As a c-ideal it minimally contains all existing joins in Θ of subsets of itself, *i.e.* the following set is contained in the join of $\{I_b\}_{b \in B}$

$$\langle \{I_b\}_{b \in B} \rangle := \{\bigvee X \mid X \subseteq \bigcup_{b \in B} I_b,\ \bigvee X \text{ exists in } \Theta\}.$$

We will show that $\langle \{I_b\}_{b \in B} \rangle$ is itself a c-ideal and thus that $\bigvee_{b \in B} I_b = \langle \{I_b\}_{b \in B} \rangle$.

Firstly, we have $\bot = \bigvee \emptyset \in \langle \{I_b\}_{b \in B} \rangle$. Now suppose we have $\bigvee X \in \langle \{I_b\}_{b \in B} \rangle$ and $y \in \Theta$ such that $y \leq \bigvee X$. If $X = \emptyset$, then $y = \bot \in \langle \{I_b\}_{b \in B} \rangle$. Otherwise, there is an $x \in X$ with $y \leq x$. Then for some $b \in B$ we have $y \in I_b$ and thus $y = \bigvee \{y\} \in \langle \{I_b\}_{b \in B} \rangle$, as required. Finally, suppose that $Y \subseteq \langle \{I_b\}_{b \in B} \rangle$ such that $\bigvee Y$ exists in Θ. We have that $Y = \{\bigvee X_c\}_{c \in C}$ for some $X_c \subseteq \bigcup_{b \in B} I_b$ such that $\bigvee X_c$ exists in Θ. Take $Z = \bigcup_{c \in C} X_c$. Then $Z \subseteq \bigcup_{b \in B} I_b$ and $\bigvee Z = \bigvee Y$ and therefore exists in Θ. We conclude that $\bigvee Y = \bigvee Z \in \langle \{I_b\}_{b \in B} \rangle$. □

(c) (*1 point*) Use parts (a) and (b) to conclude that Ω is a cHa.

Solution. Having proven part (b) we only have to show that Ω satisfies D in order to apply part (b) to obtain the required result. To this end, let $I \in \Omega$ and $\{I_b\}_{b\in B} \subseteq \Omega$. The required equality is found by

$$\begin{aligned}
I \wedge \bigvee_{b\in B} I_b &= I \cap \{\bigvee X \mid X \subseteq \bigcup_{b\in B} I_b,\ \bigvee X \text{ exists in } \Theta\} \\
&= \{\bigvee X \mid X \subseteq \bigcup_{b\in B} I_b,\ \bigvee X \text{ exists in } \Theta,\ \bigvee X \in I\} \\
&= \{\bigvee X \mid X \subseteq \bigcup_{b\in B} I_b,\ \bigvee X \text{ exists in } \Theta,\ X \in I\} \\
&= \{\bigvee X \mid X \subseteq \bigcup_{b\in B} (I_b \cap I),\ \bigvee X \text{ exists in } \Theta\} = \bigvee_{b\in B} (I_b \wedge I),
\end{aligned}$$

where the third equality holds from left to right due to the fact that I is downwards closed and from right to left due to the fact that it is closed under existing joins. $\square$

(d) (*1 point*) Let $i : \Theta \to \Omega$ be the function given by $i(x) = \{y \in \Theta \mid y \leq x\}$. Show that i is an embedding, *i.e.* that it is injective and preserves $\to, \perp$ and existing $\bigvee, \bigwedge$. This means, for instance, that

$$i(\bigvee a) = \bigvee i(a).$$

Solution. For injectivity, let $x, y \in \Theta$ such that $x \neq y$. Suppose w.l.o.g. that $y \not\leq x$. Then $x \in i(x)$ but $y \notin i(x)$ and thus $i(x) \neq i(y)$, as required.

In showing the preservation of operations, let us start with $\bigwedge$. For $X \subseteq \Theta$ such that $\bigwedge X$ exists in Θ we have

$$\begin{aligned}
i(\bigwedge X) &= \{y \in \Theta \mid y \leq \bigwedge X\} \\
&= \bigcap_{x\in X} \{y \in \Theta \mid y \leq x\} \\
&= \bigwedge_{x\in X} i(x),
\end{aligned}$$

as required.

On to $\bigvee$. Let $X \subseteq \Theta$ such that $\bigvee X$ exists. First suppose that $y \in i(\bigvee X)$, *i.e.* that $y \leq \bigvee X$. Since $X \subseteq \bigcup_{x\in X} i(x)$ and, as we have seen, also $\bigvee_{x\in X} i(x)$ as a c-ideal is downwards closed, we have $y \in \bigvee_{x\in X} i(x)$. For the converse, let $\bigvee Y \in \bigvee_{x\in X} i(x)$. For each $y \in Y$, we have for some $x \in X$ that $y \in i(x)$ and thus that $y \leq x$. This means that $\bigvee Y \leq \bigvee X$, which implies $\bigvee Y \in i(\bigvee X)$, as required.

We will now show that i preserves $\to$. We will do this by showing that $i(x \to y)$ is the greatest element of

$$H := \{I \in \Omega \mid I \wedge i(x) \leq i(y)\},$$

from which it follows that $i(x \to y) = i(x) \to i(y)$. First let $z \in i(x \to y) \cap i(x)$. It then holds that $z = z \wedge z \leq z \wedge x \leq y$, where the first inequality obtains due to the fact that $z \in i(x)$ and the second due to the fact that $z \in i(x \to y)$. We conclude that $z \in i(y)$ and, since this holds for arbitrary $z \in i(x)$, that $i(x) \subseteq i(y)$. This means that $i(x \to y) \in H$.

To see that it is also the greatest element of H, let $I \in H$ be arbitrary. We wish to show that $I \subseteq i(x \to y)$. To this end, let $z \in I$. We have $z \wedge x \leq z$, so by the fact that I is downwards closed, also $z \wedge x \in I$. It further holds that $z \wedge x \leq x$ and thus that $z \wedge x \in i(x)$. Since $I \in H$, we find $z \wedge x \leq y$, *i.e.* $z \in i(x \to y)$, as required.

We finish the proof by noting that $i(\perp) = i(\bigvee \emptyset) = \bigvee_{x\in\emptyset} i(x) = \perp$. $\square$