Seminar on Models of Intuitionism

Hand-in exercise 4: model solution

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Exercise 1. (a) For all $q \in \mathbb{Q}$, we have $[\![q = q]\!] = T$, so $[\![q \in \mathbb{Q}]\!] = T$. Furthermore, for every $\xi, \eta \in \mathcal{R}$, we have $[\![A(\xi)]\!] \cap [\![\xi = \eta]\!] \subseteq [\![A(\eta)]\!]$. This gives:

$$\begin{aligned}\bigcup_{q\in\mathbb{Q}} [\![A(q)]\!] &= \bigcup_{q\in\mathbb{Q}} ([\![q \in \mathbb{Q}]\!] \cap [\![A(q)]\!]) \subseteq \bigcup_{\xi\in\mathbb{R}} ([\![\xi \in \mathbb{Q}]\!] \cap [\![A(\xi)]\!]) = \bigcup_{\xi\in\mathbb{R}} \left(\left(\bigcup_{q\in\mathbb{Q}} [\![\xi = q]\!] \right) \cap [\![A(\xi)]\!] \right) \\ &= \bigcup_{\xi\in\mathcal{R}, q\in\mathbb{Q}} ([\![\xi = q]\!] \cap [\![A(\xi)]\!]) \subseteq \bigcup_{\xi\in\mathcal{R}, q\in\mathbb{Q}} [\![A(q)]\!] = \bigcup_{q\in\mathbb{Q}} [\![A(q)]\!].\end{aligned}$$

We conclude that $[\![\exists x (x \in \mathbb{Q} \wedge A(x))]\!] = \bigcup_{\xi\in\mathbb{R}} ([\![\xi \in \mathbb{Q}]\!] \cap [\![A(\xi)]\!]) = \bigcup_{q\in\mathbb{Q}} [\![A(q)]\!]$. $\square$

(b) Let $\xi, \eta \in \mathbb{R}$. We notice that $[\![\forall z z' (\xi < z < \eta \wedge z = z' \rightarrow \xi < z' < \eta)]\!] = T$. Indeed, if $\theta, \theta' \in \mathcal{R}$ and $t \in T$ satisfy $\xi(t) < \theta(t) < \eta(t)$ and $\theta(t) = \theta(t')$, then we also have $\xi(t) < \theta'(t) < \eta(t)$.

Now let $t \in T$ such that $\xi(t) < \eta(t)$. According exercise (a), we should prove that

$$t \in [\![\exists z (z \in \mathbb{Q} \wedge (\xi < z < \eta))]\!] = \bigcup_{q\in\mathbb{Q}} [\![\xi < q < \eta]\!] = \bigcup_{q\in\mathbb{Q}} \{s \in T \colon \xi(s) < q < \eta(s)\}.$$

But this indeed holds, since there exists a rational number q such that $\xi(t) < q < \eta(t)$. $\square$

(c) Let $\underline{0}$ denote the infinite zero sequence. Define the clopen sets

$$U_n = \{t \in T \colon \bar{t}(n) = \overline{\underline{0}}(n) \text{ and } t(n) \neq 0\}.$$

Define ξ by $\xi(t) = \frac{1}{n+1}$ if $t \in U_n$ and $\xi(\underline{0}) = 0$. We check that ξ is continuous. Since all the U_n are clopen, we only need to check continuity at $\underline{0}$. Let $\varepsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \varepsilon$. Then $\underline{0}$ is an element of the open set $U = \{t \in T \colon \bar{t}(n) = \overline{\underline{0}}(n)\} = \{\underline{0}\} \cup \bigcup_{m\geq n} U_m$, and for all $t \in U$, we have $|\xi(t) - \xi(\underline{0})| = |\xi(t)| \leq \frac{1}{n+1} < \varepsilon$, as desired.
Notice that $\mathrm{Int}\{t \in T \mid \xi(t) \in \mathbb{Q}\} = \mathrm{Int}(T) = T$. On the other hand, $\xi(\underline{0}) = 0$, but $\underline{0}$ is not an element of $[\![\xi = 0]\!] = \mathrm{Int}\{t \in T \colon \xi(t) = 0\} = \mathrm{Int}\{\underline{0}\} = \emptyset$. So $\underline{0} \notin \bigcup_{q\in\mathbb{Q}} [\![\xi = q]\!] = [\![\xi \in \mathbb{Q}]\!]$. $\square$

Exercise 2. (a) Since Φ is continuous, the function $a \mapsto |\Phi(s, \xi(s)) - \Phi(s, a)|$ from $\mathbb{R}$ to $\mathbb{R}$ is also continuous. The function α takes the supremum of this function over the nonempty interval $[\xi(s) - \delta, \xi(s) + \delta]$. This interval is compact, so this supremum exists, and the bounderies of this interval depend in a continuous way on s and δ (since ξ is continuous), so α is itself continuous. For fixed s, the function $a \mapsto \Phi(s, a)$ is also continuous. This means that $\lim_{a \to \xi(s)} |\Phi(s, \xi(s)) - \Phi(s, a)| = 0$, which gives the second statement. $\square$

(b) Fix $\delta > 0$ such that $\alpha(t, \delta) < \varepsilon(t)$. Since the function $T \to \mathbb{R}$ given by $s \mapsto \alpha(s, \delta) - \varepsilon(s)$ is continuous, there is an open $U \subseteq T$ such that $t \in U$ and $\alpha(s, \delta) < \varepsilon(s)$ for all $s \in U$. We will show that

$$U \subseteq \bigcap_{\eta \in \mathcal{R}} (\operatorname{Int}\{s \in T \colon |\xi(s) - \eta(s)| \geq \delta\} \cup \{s \in T \colon |\Phi(s, \xi(s)) - \Phi(s, \eta(s))| < \varepsilon(s)\}).$$

This suffices to prove the result, since we have $\operatorname{Int}(A) \cup B \subseteq \operatorname{Int}(A \cup B)$ for every $A \subseteq T$ and *open* $B \subseteq T$. Let $s \in U$ and $\eta \in \mathcal{R}$. If $|\xi(s) - \eta(s)| > \delta$, then

$$s \in \{s \in T \colon |\xi(s) - \eta(s)| > \delta\} \subseteq \operatorname{Int}\{s \in T \colon |\xi(s) - \eta(s)| \geq \delta\}.$$

On the other hand, if $|\xi(s) - \eta(s)| \leq \delta$, then by the definition of α, we get

$$|\Phi(s, \xi(s)) - \Phi(s, \eta(s))| \leq \alpha(s, \delta) < \varepsilon(s).$$

In both cases, we are done. $\square$

(c) Let $\varphi \in \mathcal{R}^{\mathcal{R}}$ and $\xi, \varepsilon \in \mathcal{R}$ be given. Consider $t \in T$ such that $\varepsilon(t) > 0$ and take a δ as in exercise (b). For an $\eta \in \mathcal{R}$, we have $[\![\xi - \delta < \eta < \xi + \delta]\!] = \{s \in T \colon |\xi(s) - \eta(x)| < \delta\}$. Furthermore:

$$\begin{aligned} [\![\varphi(\xi) - \varepsilon < \varphi(\eta) < \varphi(\xi) + \varepsilon]\!] &= \{s \in T \colon |\varphi(\xi)(s) - \varphi(\eta)(s)| < \varepsilon(s)\} \\ &= \{s \in T \colon |\Phi(s, \xi(s)) - \Phi(s, \eta(s))| < \varepsilon(s)\} \end{aligned}$$

This means that

$$\begin{aligned} &[\![\forall y \ (\xi - \delta < y < \xi + \delta \to \varphi(\xi) - \varepsilon < \varphi(y) < \varphi(\xi) + \varepsilon)]\!] \\ &= \operatorname{Int} \bigcap_{\eta \in \mathcal{R}} (\operatorname{Int}(T \backslash [\![\xi - \delta < \eta < \xi + \delta]\!]) \cup [\![\varphi(\xi) - \varepsilon < \varphi(\eta) < \varphi(\xi) + \varepsilon]\!]) \\ &= \operatorname{Int} \bigcap_{\eta \in \mathcal{R}} (\operatorname{Int}\{s \in T \colon |\xi(s) - \eta(s)| \geq \delta\} \cup \{s \in T \colon |\Phi(s, \xi(s)) - \Phi(s, \eta(s))| < \varepsilon(s)\}). \end{aligned}$$

Since $[\![\delta > 0]\!] = T$, we can conclude from exercise (b) that

$$t \in [\![\delta > 0 \wedge \forall y \ (\xi - \delta < y < \xi + \delta \to \varphi(\xi) - \varepsilon < \varphi(y) < \varphi(\xi) + \varepsilon)]\!],$$

and thus that

$$[\![\varepsilon > 0 \to \exists \delta (\delta > 0 \wedge \forall y \ (\xi - \delta < y < \xi + \delta \to \varphi(\xi) - \varepsilon < \varphi(y) < \varphi(\xi) + \varepsilon))]\!] = T,$$

which proves the result. $\square$

Marking scheme.

1(a) There where two inclusions to prove, both worth half a point. For saying that $[\![\forall x \forall y \phi(x,y)]\!]$ is the same as $\mathrm{Int} \bigcap_{x,y\in\mathcal{R}} [\![\phi(x,y)]\!]$, half a point was deducted. If the same mistake was made again in exercise b, no points where deducted for that exercise.

1(b) Correctly showing extensionality of $\xi < z < \eta$ was worth 1 point. Using exercise (a) in the correct way was worth half a point, and drawing the correct conclusion was also worth half a point.

1(c) Half a point was awarded for giving a correct function. Half a point for proving that it is continuous (which could be done in many different ways), and one point for proving that it has the required property.

2(a) This exercise had three parts: the well-definedness of α, the continuity of α, and the behaviour when $\delta \to 0$. For missing one of these parts, $\frac{1}{2}$ point was deducted (with a minimum score of 0, of course). For the continuity of α, it was important to say something about the boundaries of the compact interval over which the supremum was taken, e.g. that they depend in a continuous manner on s and δ.

2(b) One point was awarded for the idea of considering an open around t on which $\alpha(s,\delta) < \varepsilon(s)$ holds, and one point was awarded for finishing the argument. This means that it was in principle possible to get one point if you didn't know how to handle the interiors, but could give the rest of the argument (e.g. by solving a version of the exercise without the interiors). In practice, this didn't happen.

2(c) One point was awarded for recognizing the set from exercise (b) as the interpretation of $\forall y\ (\xi - \delta < y < \xi + \delta \to \varphi(\xi) - \varepsilon < \varphi(y) < \varphi(\xi) + \varepsilon)$. Here it was important to mention the rule $\varphi(\xi)(t) = \Phi(t, \xi(t))$. The other point was awarded for finishing the argument.