Seminar on Models of Intuitionism

Solutions to hand-in exercise 9

4 May

Exercise 1.

(a) Let $f, g\colon \mathbb{N} \to \mathbb{N}$. First of all, note that we can define f as $\lambda y.(f \oplus g)(2y)$, so $f \leq_T f \oplus g$. Similarly, $g \leq_T f \oplus g$, so $f \oplus g$ is indeed an upper bound. Now let $h\colon \mathbb{N} \to \mathbb{N}$ be such that $f, g \leq_T h$. We must show that $f \oplus g \leq_T h$. Note we can define $f \oplus g$ in terms of f and g and by checking whether the input is even or odd. Clearly, the latter is recursive. Since both f and g are h-recursive, we find that $f \oplus g$ is h-recursive, as desired. Explicitly, we may define $f \oplus g$ as

$$\lambda x.f(\mu y < x[x = 2y]) \cdot [(\mu y < x[x = 2y]) = x] + g(\mu y < x[x = 2y + 1]) \cdot [(\mu y < x[x = 2y + 1]) = x].$$

■

Onoe point for showing that $f \oplus g$ is indeed an upper bound of f and g. One point for showing that it is the least. This does not have to be done explicitly, but the student should mention that the case distinction is recursive and that f and g are both h-recursive.

(b) Suppose we have mass problems $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$ with $\mathcal{A} \equiv_w \mathcal{A}'$ and $\mathcal{B} \equiv_w \mathcal{B}'$. We show that $\mathcal{A} \to \mathcal{B} \leq_w \mathcal{A}' \to \mathcal{B}'$ and note that the converse is proved similarly. Let $f \in \mathcal{A}' \to \mathcal{B}'$. We claim that $f \in \mathcal{A} \to \mathcal{B}$. Let $g \in \mathcal{A}$. Since $\mathcal{A} \equiv_w \mathcal{A}'$, we have $g' \in \mathcal{A}'$ with $g' \leq_T g$. As $f \in \mathcal{A}' \to \mathcal{B}'$, we get $h' \in \mathcal{B}'$ with $h' \leq_T f \oplus g'$. Since $\mathcal{B} \equiv_w \mathcal{B}'$, we obtain $h \in \mathcal{B}$ with $h \leq_T h' \leq_T f \oplus g'$. But $g' \leq_T g$, so $f, g' \leq_T f \oplus g'$. Thus, $f \oplus g' \leq_T f \oplus g$. Hence, $h \leq_T f \oplus g$, as desired. ■

(c) Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be mass problems. Suppose we have $\mathcal{A} \vee \mathcal{C} \geq_w \mathcal{B}$. That is, $C(\mathcal{A}) \cap C(\mathcal{C}) \geq_w \mathcal{B}$. We show that $\mathcal{C} \geq_w \mathcal{A} \to \mathcal{B}$. Let $f \in \mathcal{C}$. We claim that $f \in \mathcal{A} \to \mathcal{B}$. Let $g \in \mathcal{A}$. Then $f \oplus g \in C(\mathcal{A}) \cap C(\mathcal{C})$, by part (a) and since $C(\mathcal{A})$ and $C(\mathcal{C})$ are upwards closed w.r.t. $\leq_T$. By assumption, we now get $h \in \mathcal{B}$ with $h \leq_T f \oplus g$. Hence, $f \in \mathcal{A} \to \mathcal{B}$.

Conversely, assume $\mathcal{C} \geq_w \mathcal{A} \to \mathcal{B}$. Let $f \in C(\mathcal{A}) \cap C(\mathcal{C})$, i.e. suppose we have $g_{\mathcal{A}} \in \mathcal{A}$ and $g_{\mathcal{C}} \in \mathcal{C}$ with $g_{\mathcal{A}}, g_{\mathcal{C}} \leq_T f$. By assumption, we get $f' \in \mathcal{A} \to \mathcal{B}$ with $f' \leq_T g_{\mathcal{C}}$. Hence, we obtain $h \in \mathcal{B}$ with $h \leq_T g_{\mathcal{A}} \oplus f'$. But $f' \leq_T g_{\mathcal{C}}$, so $h \leq_T g_{\mathcal{A}} \oplus g_{\mathcal{C}}$. Finally, $g_{\mathcal{A}}, g_{\mathcal{C}} \leq_T f$, so by part (a) we have $h \leq_T f$, as desired. ■

One point for each direction.

Exercise 2.

(a) Let's first think of a necessary condition on $C(\mathcal{A})$. Suppose that $[\mathcal{A}]$ is join-reducible. Then there exist $[\mathcal{B}]$ and $[\mathcal{C}]$ such that $[\mathcal{A}] = [\mathcal{B}] \vee [\mathcal{C}] = [C(\mathcal{B}) \cap C(\mathcal{C})]$ and $[\mathcal{B}] \neq [\mathcal{A}]$ and $[\mathcal{C}] \neq [\mathcal{A}]$. So we have $\mathcal{A} \not\leq_w \mathcal{B}$ and $\mathcal{A} \not\leq_w \mathcal{C}$. So there exist $g \in \mathcal{B}$ and $h \in \mathcal{C}$ such that for all $f \in \mathcal{A}$ we have $f \not\leq_T g, h$. Hence, $g, h \notin C(\mathcal{A})$. But notice that we have $g \leq_T g \oplus h$ and $h \leq_T g \oplus h$, thus $g \oplus h \in C(\mathcal{B}) \cap C(\mathcal{C})$. Since $[\mathcal{A}] = [\mathcal{B}] \vee [\mathcal{C}]$, there is some $f \in \mathcal{A}$ such that $f \leq_T g \oplus h$, so we see that $g \oplus h \in C(\mathcal{A})$. We now formulate the condition:

$$[\mathcal{A}] \text{ is join-reducible iff } \exists g, h \notin C(\mathcal{A}) \text{ with } g \oplus h \in C(\mathcal{A}).$$

Proof
(only if): Using the argument above we find such g and h.
(if): Suppose there exists $g, h \notin C(\mathcal{A})$ with $g \oplus h \in C(\mathcal{A})$. We claim that $[\mathcal{A}] = [\mathcal{A} \cup \{g\}] \vee [\mathcal{A} \cup \{h\}]$, i.e. $\mathcal{A} \equiv_w C(\mathcal{A} \cup \{g\}) \cap C(\mathcal{A} \cup \{h\})$. Certainly, the $\geq_w$–inequality holds, as any $f \in \mathcal{A}$ is also in $C(\mathcal{A} \cup \{g\}) \cap C(\mathcal{A} \cup \{h\})$. Now let $f \in C(\mathcal{A} \cup \{g\}) \cap C(\mathcal{A} \cup \{h\})$. Then there exist $f_0 \in \mathcal{A} \cup \{g\}$ and $f_1 \in \mathcal{A} \cup \{h\}$ such that $f_0, f_1 \leq_T f$. If either of $f_i \in \mathcal{A}$ we are done. Suppose that $f_0 = g$ and $f_1 = h$. Then by exercise 1(a) we have $g \oplus h \leq_T f$. By assumption $g \oplus h \in C(\mathcal{A})$, so there exists $f' \in \mathcal{A}$ with $f' \leq_T g \oplus h \leq_T f$. By transitivity, we have the desired $f' \leq_T f$. Secondly, note that from $g, h \notin C(\mathcal{A})$ it follows that $\mathcal{A} \not\leq_T C(\mathcal{A} \cup \{g\})$ and $\mathcal{A} \not\leq_T C(\mathcal{A} \cup \{h\})$. So $[\mathcal{A}]$ is indeed join-reducible. ■

One point for the right condition. One point for necessity and sufficiency each.

(b) Suppose that $[\mathcal{A}]$ and $[\mathcal{B}]$ are join-irreducible and $[\mathcal{A}] \wedge [\mathcal{B}] = [\mathcal{A} \cup \mathcal{B}]$ is join-reducible. Hence, by part (a) we have $g, h \notin C(\mathcal{A} \cup \mathcal{B})$ with $g \oplus h \in C(\mathcal{A} \cup \mathcal{B})$. Hence there is some $f \in \mathcal{A} \cup \mathcal{B}$ with $f \leq_T g \oplus h$. Suppose w.l.o.g. that $f \in A$. Then $g \oplus h \in C(\mathcal{A})$. Hence, as $[\mathcal{A}]$ is join-irreducible we must have either $g \in C(\mathcal{A})$ or $h \in C(\mathcal{A})$. Suppose w.l.o.g. that $g \in C(\mathcal{A})$. But then clearly also $g \in C(\mathcal{A} \cup \mathcal{B})$, which is a contradiction. Hence $[\mathcal{A}] \wedge [\mathcal{B}] = [\mathcal{A} \cup \mathcal{B}]$ must also be join-irreducible. We conclude that $\mathfrak{M}_w$ is not dd-like.

One point for using part (a) to find such g and h. One point for observing that $g \in C(\mathcal{A})$ or $h \in C(\mathcal{A})$ and completing the proof.