1 Introduction

In constructive mathematics one finds two different formalizations of Church's thesis:

$$\forall x \exists y A(x,y) \to \exists z \forall x \exists v (Tzxv \wedge A(x, U(v))) \tag{CT$_0$}$$

$$\forall x \exists! y A(x,y) \to \exists z \forall x \exists v (Tzxv \wedge A(x, U(v))) \tag{CT$_0$!}$$

Today's goal is to prove a result of Lifschitz (1979) stating that $\mathbf{HA} + \mathrm{CT}_0$ is stronger than $\mathbf{HA} + \mathrm{CT}_0!$ To this end, we will meet a new flavor of realizability.

Let $j : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a paring function. Denote the first and second components of a number n as $j_1(n)$ and $j_2(n)$ respectively.

Definition 1.1. For every $e \in \mathbb{N}$, define $V_e = \{n \leq j_2(e) : \varphi_{j_1(e)}(n) \uparrow\}$.

Definition 1.2. Let $e \in \mathbb{N}$. Then:

- e realizes $t = s$ iff $t = s$,
- e realizes $A \wedge B$ iff $j_1(e)$ realizes A and $j_2(e)$ realizes B,
- e realizes $A \to B$ iff for every n realizing A, $\varphi_e(n)$ is defined and $\varphi_e(n)$ realizes B,
- e realizes $\forall x A(x)$ iff for all n: $\varphi_e(n)$ is defined and $\varphi_e(n)$ realizes $A(n)$,
- e realizes $\exists x A(x)$ iff V_e is non-empty and for every $n \in V_e$, $j_2(n)$ realizes $A(j_1(n))$.

2 Preliminaries

Definition 2.1. A set S of natural numbers is called *recursively enumerable*, or simply *r.e.*, if there exists a partial recursive function f such that

$$f(x) = \begin{cases} 0, & \text{if } x \in S \\ \uparrow, & \text{otherwise} \end{cases}$$

Proposition 2.2. *The halting set $H = \{(i,x) : \varphi_i(x) \downarrow\}$ is recursively enumerable.*

Definition 2.3. Two disjoint sets A and B of natural numbers are called *recursively inseparable* if there exists no recursive set C such that $A \subseteq C$ and $B \subseteq \mathbb{N} \backslash C$.

3 Preparatory Lemma's

The following lemma's correspond to lemma's 1–5 of Lifschitz (1979).

Lemma 3.1. *There exists a unary partial recursive function α such that for every e, we have $|V_e| = 1$ implies $\alpha(e)$ is defined and $\alpha(e) \in V_e$.*

Lemma 3.2. *There exists a unary total recursive function β such that for every n, it holds that $V_{\beta(n)} = \{n\}$.*

Lemma 3.3. *There exists a unary total recursive function γ such that for every e, it holds that $V_{\gamma(e)} = \bigcup_{n \in V_e} V_n$.*

Lemma 3.4. *For every unary partial recursive function θ there exists a unary partial recursive function θ^* such that for every e, we have $V_e \subseteq dom\theta$ implies $\theta^*(e)$ is defined and $V_{\theta^*(e)} = \theta(V_e)$*

Lemma 3.5. *For every formula A there exists a unary partial recursive function ϕ_A such that for any non-empty V_e, if every element of V_e realizes a closed instance $\bar{A}$ of A, then $\phi_A(e)$ is defined and realizes $\bar{A}$.*

4 Main Results

Lemma 4.1. *Every theorem of $\boldsymbol{HA} + \mathrm{CT}_0!$ is realizable*

Definition 4.2. The binary version of CT_0 is:

$$\forall x(A(x) \vee B(x)) \longrightarrow \exists z \forall x \exists v (T(z,x,v) \wedge (U(v) = 0 \longrightarrow A(x)) \wedge (U(v) \neq 0 \longrightarrow B(x))$$

Theorem 4.3. *There exists a closed instance of CT_0^b which is underivable in $\boldsymbol{HA} + \mathrm{CT}_0!$*

Corollary 4.4. *There exists a closed instance of CT_0 which is underivable in $\boldsymbol{HA} + \mathrm{CT}_0!$*

References

[1] Lifschitz, Vladimir. (1979). "CT_0 is stronger than $\mathrm{CT}_0!$" *Proceedings of the American Mathematical Society* 73 (1), 101–106.