# Seminar on Models of Intuitionism

Hand-out lecture 11

11 May

## 1 Connectionally Closed Categories

**Definition 1.1.** A *category* $\mathcal{C}$ consists of a collection $\mathcal{C}_0$ of *objects* and a collection $\mathcal{C}_1$ of *arrows* (or *morphisms*) such that the following holds.

- Each arrow has a *domain* and a *codomain* which are objects; one writes $f\colon A \to B$ or $A \xrightarrow{f} B$ if $A$ is the domain of the arrow $f$ and $B$ is its codomain.
- Given two arrows $A \xrightarrow{f} B \xrightarrow{g} C$, there is a composition $A \xrightarrow{g\circ f} C$ and composition is associative.
- For every object $A$ there is an *identity arrow* $1_A\colon A \to A$, satisfying $1_A \circ g = g$ for every $g\colon B \to A$ and $f \circ 1_A = f$ for every $f\colon A \to B$.

Equationally,

$$f \circ 1_A = f \qquad \text{for any } f\colon A \to B; \tag{1}$$

$$1_B \circ g = g \qquad \text{for any } g\colon A \to B; \tag{2}$$

$$(h \circ g) \circ f = h \circ (g \circ f) \qquad \text{for any } f\colon A \to B, g\colon B \to C \text{ and } h\colon C \to D. \tag{3}$$

**Definition 1.2.** A morphism $f\colon A \to B$ in $\mathcal{C}$ is an *isomorphism* if there is a morphism $g\colon B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

**Example 1.3.** We have a category $\mathsf{Set}$ whose objects are sets and the arrows are functions between sets. Composition is ordinary function composition.

**Example 1.4.** Let $(P, \leq)$ be a poset. We view $P$ as a category whose objects are the elements of $P$ and we have an arrow $p \to q$ iff $p \leq q$. Observe that arrows are unique in this category and that all isomorphisms are identities.

### 1.1 Categorical Constructions

From now on we will work in some fixed category $\mathcal{C}$ and $(P, \leq)$ will always denote some fixed poset.

**Definition 1.5.** A *terminal object* $T$ in $\mathcal{C}$ is an object such that there is exactly one arrow $A \to T$ for any object $A$ in $\mathcal{C}$.

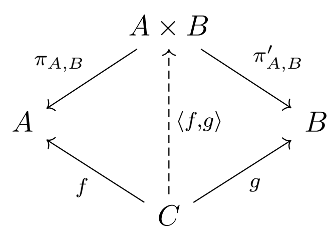
**Example 1.6.** Singleton sets are terminal objects in the category $\mathsf{Set}$.

**Example 1.7.** Note that $p \in P$ is terminal iff $p$ is the greatest element of $P$.

Note that in general, a terminal object is unique up to isomorphism. (This is why category theorists usually speak of *the* terminal object in a category.) We will make an explicit choice and write $t$ for the chosen terminal object and $!_A \colon A \to t$ for the unique arrow from $A$ to $t$. We have the following equation

$$f = !_A \quad \text{for any } f \colon A \to t. \tag{4}$$

**Definition 1.8.** A *product* of two objects $A$ and $B$ is an object $A \times B$ with morphisms $\pi_{A,B} \colon A \times B \to A$ and $\pi'_{A,B} \colon A \times B \to B$ (called *projections*) such that for any $f \colon C \to A$ and $g \colon C \to B$ there is a unique arrow $\langle f, g \rangle \colon C \to A \times B$ making the following diagram commute




**Definition 1.9.** We say that $\mathcal{C}$ *has (binary) products* if a product of $A$ and $B$ exists for each pair of objects $A$ and $B$ in $\mathcal{C}$.

**Example 1.10.** The cartesian product of two sets (with obvious projection maps) is a product in $\mathsf{Set}$.

**Example 1.11.** The product of $p, q \in P$ in $P$ is the greatest lower bound (with respect to $\leq$) of $p$ and $q$. So we see that $P$ has products iff (binary) meets exist in $P$.

Again, for two given fixed objects $A$ and $B$ a product of $A$ and $B$ is unique up to isomorphism. For each pair of objects $A$ and $B$ we will specify a product $A \times B$ together with projections $\pi_{A \times B}$ and $\pi'_{A \times B}$. The defining equations (with ommited subscripts) read

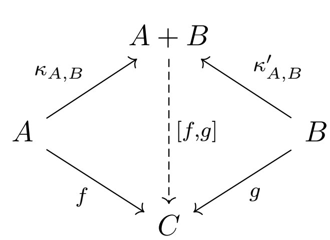
$$\pi \circ \langle f, g \rangle = f \qquad \text{for any } f \colon C \to A \text{ and } g \colon C \to B; \tag{5}$$

$$\pi' \circ \langle f, g \rangle = g \qquad \text{for any } f \colon C \to A \text{ and } g \colon C \to B; \tag{6}$$

$$\langle \pi \circ h, \pi' \circ h \rangle = h \qquad \text{for any } h \colon C \to A \times B. \tag{7}$$

Similarly (or really, dually), we have the notion of a *coproduct*. (This is a product in $\mathcal{C}^{\mathrm{op}}$.)

**Definition 1.12.** A *coproduct* of two objects $A$ and $B$ is an object $A + B$ with morphisms $\kappa_{A,B} \colon A \to A + B$ and $\kappa'_{A,B} \colon B \to A + B$ such that for any $f \colon A \to C$ and $g \colon B \to C$ there is a unique arrow $[f, g] \colon A + B \to C$ making the following diagram commute




**Definition 1.13.** The category $\mathcal{C}$ is said to *have (binary) coproducts* if a coproduct of $A$ and $B$ exists in $\mathcal{C}$ for any pair of objects $A$ and $B$ in $\mathcal{C}$.

**Example 1.14.** In Set, a coproduct of two sets is their disjoint union with the obvious inclusions.

**Example 1.15.** The coproduct of $p, q \in P$ in $P$ is the least upper bound (with respect to $\leq$) of $p$ and $q$. So $P$ has coproducts iff $P$ has joins. Furthermore, we see that $P$ is a lattice iff $P$ has products and coproducts.

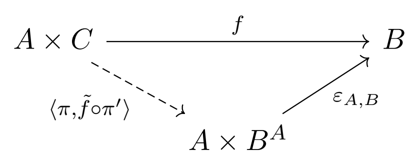
Again, we specify for each pair of objects $A$ and $B$ a coproduct $A+B$ together with morphisms $\kappa_{A,B}$ and $\kappa'_{A,B}$. The defining equations are

$$[f, g] \circ \kappa = f \quad \text{for any } f\colon A \to C \text{ and } g\colon B \to C; \tag{8}$$

$$[f, g] \circ \kappa' = g \quad \text{for any } f\colon A \to C \text{ and } g\colon B \to C; \tag{9}$$

$$[h \circ \kappa, h \circ \kappa'] = h \quad \text{for any } h\colon A + B \to C. \tag{10}$$

**Definition 1.16.** Assume that our fixed category $\mathcal{C}$ has products. An *exponential* of two objects $A$ and $B$ is an object $B^A$ with an arrow $\varepsilon_{A,B}\colon A \times B^A \to B$ such that for any $f\colon A \times C \to B$ there is a unique arrow $\tilde{f}\colon C \to B^A$ making the following diagram commute




**Example 1.17.** In Set, given two sets $X$ and $Y$, the set $Y^X$ of all functions from $X$ to $Y$ is an exponential of $X$ and $Y$. The evaluation arrow $\varepsilon_{X,Y}\colon X \times Y^X \to Y$ is given by $(x, g) \mapsto g(x)$. Further, given $f\colon X \times Z \to Y$, one may construct $\tilde{f}\colon Z \to Y^X$ by $z \mapsto (x \mapsto f(x, z))$.

**Example 1.18.** For $p, q, r \in P$, we see that the exponential $q^p$ of $p$ and $q$ should satisfy $p \wedge r \leq q$ iff $r \leq q^p$. Hence, if $P$ is a Heyting algebra, then $p \to q$ is the exponential of $p$ and $q$.

Assuming we have specified products in $\mathcal{C}$, we specify for each pair of objects $A$ and $B$ an exponential $B^A$ together with an evaluation morphism $\varepsilon_{A,B}$ satisfying the equations

$$\varepsilon \circ \langle \pi, \tilde{h} \circ \pi' \rangle = h \qquad \text{for any } h\colon A \times C \to B; \tag{11}$$

$$(\varepsilon \circ \langle \pi, k \circ \pi' \rangle)^{\sim} = k \quad \text{for any } k\colon C \to B^A. \tag{12}$$

## 1.2 Connectionally Closed Categories

**Definition 1.19.** A category is called *cartesian closed* if it has a terminal object, binary products and exponentials. We call a category *connectionally closed* (c.c.) if it is cartesian closed and has binary coproducts.

**Example 1.20.** The category Set is c.c. as is any Heyting algebra $H$ (seen as a category). In the homework, you will see another example of a c.c. category. This category will play an important role next week.

**Definition 1.21.** A *functor* $F$ between categories $\mathcal{C}$ and $\mathcal{D}$ consists of operations $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1\colon \mathcal{C}_1 \to \mathcal{D}_1$ such that for each arrow $f\colon A \to B$ in $\mathcal{C}$ we have $F_1(f)\colon F_0(A) \to F_0(B)$. Furthermore $F$ should respect composition and identities, i.e.

- for $A \xrightarrow{f} B \xrightarrow{g} C$, we have $F_1(g \circ f) = F_1(g) \circ F_1(f)$;
- for every object $A$ in $\mathcal{C}$ we have $F_1(1_A) = 1_{F_0(A)}$.

We usually just write $F$ instead of $F_0$ and $F_1$.

**Definition 1.22.** A functor between two c.c. categories is called a *c.c. functor* if it preserves terminal objects, binary (co)products and exponentials (e.g. the functor takes a product diagram to a product diagram). Such a functor is called a *c.c. morphism* if we have specified operations in both categories and $F$ preserves our specified terminal object, specified binary (co)products and specified exponentials (e.g. the functor takes our specific terminal object to the chosen terminal object in the target category).

## 2 Category of Proofs

In the following:

- $\mathcal{L}$ is a set of propositional atoms.
- *Formulae* are built from $\mathcal{L}$ using $\top$, $\wedge$, $\vee$ and $\to$ (but *not* $\bot$).
- An *entailment* is an expression of the form $A \Rightarrow B$, where $A$ and $B$ are formulae.
- A *theory* $T$ is a set of entailments.

Given a theory $T$, we can build *deductions* using the following rules. As indicated on the right, every deduction is assigned a unique term.

| Rule | Term |
|---|---|
| $\dfrac{}{A \Rightarrow A}$ TAUT | $1_A$ |
| $\dfrac{A \overset{f}{\Rightarrow} B \qquad B \overset{g}{\Rightarrow} C}{A \Rightarrow C}$ CUT | $g \circ f$ |
| $\dfrac{}{A \Rightarrow \top}$ TRUE | $!_A$ |
| $\dfrac{}{A \wedge B \Rightarrow A}$ ∧L1 | $\pi_{A,B}$ |
| $\dfrac{}{A \wedge B \Rightarrow B}$ ∧L2 | $\pi'_{A,B}$ |
| $\dfrac{C \overset{f}{\Rightarrow} A \qquad C \overset{g}{\Rightarrow} B}{C \Rightarrow A \wedge B}$ ∧R | $\langle f, g\rangle$ |
| $\dfrac{}{A \Rightarrow A \vee B}$ ∨R1 | $\kappa_{A,B}$ |
| $\dfrac{}{B \Rightarrow A \vee B}$ ∨R2 | $\kappa'_{A,B}$ |

| | |
|---|---|
| $\dfrac{A \overset{f}{\Rightarrow} C \qquad B \overset{g}{\Rightarrow} C}{A \vee B \Rightarrow C}\ \vee\mathrm{L}$ | $[f, g]$ |
| $\dfrac{}{A \wedge (A \to B) \Rightarrow B}\ \to\mathrm{L}$ | $\varepsilon_{A,B}$ |
| $\dfrac{A \wedge C \overset{f}{\Rightarrow} B}{C \Rightarrow A \to B}\ \to\mathrm{R}$ | $\tilde{f}$ |
| $\dfrac{}{\tau}\ \mathrm{T} \quad (\tau \in T)$ | $\tau$ |

For each of the twelve equations for a c.c. category, we identify the deductions denoted by both sides of the equations. For example, $f \circ 1_A = f$ for $f\colon A \to B$ identifies

$$\dfrac{\dfrac{}{A \Rightarrow A}\ \mathrm{TAUT} \qquad A \overset{f}{\Rightarrow} B}{A \Rightarrow B}\ \mathrm{CUT}$$

with $A \overset{f}{\Rightarrow} B$ itself.

**Definition 2.1.** (i) Two deductions are considered *equivalent* if the one can be constructed out of the other using a sequence of the above mentioned identifications.

(ii) The *category of proofs* $\mathcal{F}_{\mathcal{L}}(T)$ has

- as objects the $\mathcal{L}$-formulae;
- as arrows $A \to B$ the $T$-deductions with $A \Rightarrow B$ as conclusion, modulo equivalence.

**Proposition 2.2.** *$\mathcal{F}_{\mathcal{L}}(T)$ is a c.c. category with specified operations.*

## 2.1 Free Constructions

**Proposition 2.3.** *Suppose $\mathcal{D}$ is a c.c. category with specified operations and that for all $p \in \mathcal{L}$, an object $f(p)$ of $\mathcal{D}$ is given. Then there exists a unique c.c. morphism $F\colon \mathcal{F}_{\mathcal{L}}(\emptyset) \to \mathcal{D}$ such that $F(p) = f(p)$ for all $p \in \mathcal{L}$.*

We write $I$ for the obvious inclusion functor $\mathcal{F}_{\mathcal{L}}(\emptyset) \to \mathcal{F}_{\mathcal{L}}(T)$. We write $\tau \in T$ as $a(\tau) \Rightarrow c(\tau)$.

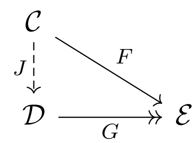
**Proposition 2.4.** *Suppose $G\colon \mathcal{F}_{\mathcal{L}}(\emptyset) \to \mathcal{D}$ is c.c. morphism, and that for all $\tau \in T$, an arrow $\hat{\tau}\colon G(a(\tau)) \to G(c(\tau))$ of $\mathcal{D}$ is given. Then there exists a unique c.c. morphism $H\colon \mathcal{F}_{\mathcal{L}}(T) \to \mathcal{D}$ such that $H(\tau) = \hat{\tau}$ for all $\tau \in T$, and $H \circ I = G$.*

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{L}}(\emptyset) & \xrightarrow{\quad G \quad} & \mathcal{D} \\ & {\scriptstyle I}\searrow \quad \nearrow{\scriptstyle H} & \\ & \mathcal{F}_{\mathcal{L}}(T) & \end{array}$$

## 2.2 Projectivity

**Definition 2.5.** (i) A c.c. morphism $G\colon \mathcal{D} \to \mathcal{E}$ is called *surjective* if $G_0$ is surjective, and for all objects $X$ and $Y$ of $\mathcal{D}$, the function $G_1\colon \mathcal{D}(X,Y) \to \mathcal{E}(G(X), G(Y))$ is surjective.

(ii) A c.c. category $\mathcal{C}$ with specified operations is called *projective* if for every surjective c.c. morphism $G\colon \mathcal{D} \to \mathcal{E}$ and every c.c. morphism $F\colon \mathcal{C} \to \mathcal{E}$, there exists a (not necessarily unique) c.c. morphism $J\colon \mathcal{C} \to \mathcal{D}$ such that $G \circ J = F$.




**Proposition 2.6.** $\mathcal{F}_{\mathcal{L}}(T)$ *is projective.*