Seminar Models of Intuisionism

Handout lecture 12: Categorical proof theory and Läuchli realizability

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Läuchli's completeness theorem: For every formula A, if $\mathbf{IQC} \nvdash A$, then there is some proof assignment p such that $p(A)$ contains no invariant functionals.

Läuchli's completeness theorem reformulated: Let $\mathcal{C}$ be a countable free c.c. category with $A, B \in \mathcal{C}_0$ such that $\mathcal{C}(A, B) = \emptyset$. Then there is a c.c. functor $F : \mathcal{C} \to \mathrm{Set}^{\mathbb{Z}}$ such that $\mathrm{Set}^{\mathbb{Z}}(F(A), F(B)) = \emptyset$.

Definition: Let $\mathcal{D}$ be a category. The poset reflection of $\mathcal{D}$ is the poset, regarded as a category, with elements equivalence classes of objects of $\mathcal{D}$ under the relation $X \sim Y$ if $\mathcal{D}(X, Y)$ and $\mathcal{D}(Y, X)$ are both nonempty. We let $[X] \leq [Y]$ if $\mathcal{D}(X, Y)$ is nonempty. We denote this poset by $Po(\mathcal{D})$.

Proof strategy: We use that the category $\mathcal{C}$ is projective. So consider the following diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad h \quad} & \\ \downarrow F & & \searrow \\ \mathrm{Set}^{\mathbb{Z}} & \xrightarrow{\quad \delta \quad} & Po(\mathrm{Set}^{\mathbb{Z}}) \end{array}$$

There is an obvious surjective functor δ, so it suffices to prove the existence of a c.c. functor h such that h preserves the emptiness of $\mathcal{C}(A, B)$. We will construct h as the following composition:

$$\mathcal{C} \longrightarrow H \longrightarrow 2^I \longrightarrow 2^{I_0} \longrightarrow 2^J \longrightarrow Po(\mathrm{Set}^{\mathbb{Z}})$$

First step

Let H be the poset reflection of $\mathcal{C}$, which we can regard as an almost Heyting algebra (aH algebra), which is a Heyting algebra but possibly without a bottom element. $\gamma : \mathcal{C} \to H$ is the obvious surjective functor.

Second step

Definition: A *filter* on a Heyting algebra K is a subset $F \subset K$ such that:

- $0 \notin F$ and $1 \in F$
- If $x \in F$ and $x \leq y$, then $y \in F$

- If $x, y \in F$ then $x \wedge y \in F$.

We call a filter F *prime* if $x \in F$ or $y \in F$ whenever $x \vee y \in F$.

Lemma: Let P be a poset. The set of all order-preserving maps $\phi : P \to 2$, denoted by 2^P, is a Heyting algebra with constant 0 function the minimal and constant 1 function the maximal element. Meets and joins are pointwise and implication is given by:

$$(\phi \to \psi)(p) = 1 \quad \Leftrightarrow \quad \forall q \geq p(\phi(q) = 1 \Rightarrow \psi(q) = 1)$$

Definition: [H,2] is the poset of all almost lattice homomorphisms from H to 2.

Proposition: Consider the evaluation mapping: $e : H \to 2^{[H,2]}$, given by $e(x)(i) = i(x)$. This is an aH algebra embedding. Hence the second step is finished by picking $I = [H, 2]$.

Third step

We let I_0 be a suitable subset of I which contains a least element, and define $\Phi : 2^I \to 2^{I_0}$ by $\Phi(\phi) = \phi|_{I_0}$.

Fourth step

Lemma: Let N, M be posets and $f : N \to M$ order-preserving. Define $f^* : 2^M \to 2^N$ by $\phi \mapsto \phi \circ f$. Then f^* is a lattice homomorphism, and f^* preserves the Heyting implication if and only if for all $p \in N$ and $\phi, \psi \in 2^M$:

$$\forall m \geq f(p)(\phi(m) = 1 \Rightarrow \psi(m) = 1) \quad \Leftrightarrow \quad \forall n \geq p(\phi(f(n)) = 1 \Rightarrow \psi(f(n)) = 1)$$

Proposition: There exists a countable subposet $J \subseteq I_0$ such that if $k : J \to I_0$ is the inclusion map, then $k^* \circ \Phi \circ e : H \to 2^J$ is an aH algebra homomorphism, and

$$k^* \circ \Phi \circ e \circ \gamma(A) \not\leq k^* \circ \Phi \circ e \circ \gamma(B)$$

Preparation for the fifth step

Proposition: $\mathrm{Set}^{\mathbb{Z}}(X, Y) \neq \emptyset$ if and only if for all $n \in \mathbb{Z}$ we have that if there is $x \in X$ such that $n \cdot x = x$, then there is $y \in Y$ such that $n \cdot y = y$.

Definition: For a $\mathbb{Z}$-set X define

$$\delta(X) = \{n \in \mathbb{N} : \exists x \in X, n \cdot x = x\}.$$

This allows an elegant reformulation of the last proposition: $\mathrm{Set}^{\mathbb{Z}}(X, Y) \neq \emptyset$ iff $\delta(X) \subseteq \delta(Y)$.

Proposition: $Po(\mathrm{Set}^{\mathbb{Z}}) \cong \mathrm{Up}(\mathbb{N}) \cong 2^{\mathbb{N}}$, where the first isomorphism is given by δ and the second isomorphism is obtained by identifying the up-sets with their characteristic functions.

Fifth step

Definition: Let P and Q be posets, we call $f : P \to Q$ *upward closed* if for all $p \in P$ we have $f(\uparrow p) = \uparrow f(p)$. If f is also surjective, we call f *quite surjective*.

Proposition: If $f : P \to Q$ is quite surjective and order preserving, then $f^* : 2^Q \to 2^P, a \mapsto a \circ f$ is an aH algebra embedding.

Proposition: Let J be a countable poset with least element and greatest element. Then there is quite surjective order preserving $f : \mathbb{N} \to J$.

Finish this step by applying the last two propositions to the poset $(\mathbb{N}, |)$ and J from step four.