Hand in exercises for section 8

To be handed in on the 7th of March

Exercise 1 We have discussed in the lecture that, for the General Reflection theorem to hold we need a list of properties. These are the following: for a hierarchy $(W_\alpha | \alpha \in \mathbf{Ord})$ with $W = \bigcup_{\alpha \in \mathbf{Ord}} W_\alpha$ we must have:

1. W_α is transitive for every α.
2. W_α is definable by a formula $\Psi(x, \alpha)$ of LST.
3. $\alpha < \beta \to W_\alpha \subseteq W_\beta$
4. $\lim(\gamma) \to W_\delta = \bigcup_{\alpha < \delta} W_\alpha$

Show for each of these properties where in the proof of the General Reflection principle they are used. These explanations will vary from pointing out a line in the proof to giving a small discussion why certain steps in the proof are allowed to be taken. We don't expect you to have written down or remembered the proof from the lecture; instead, for this exercise, refer to the proof as is written down in Devlin's book.

Exercise 2 Prove that there is a Π_1 formula expressing "x is finite" and conclude from that that "x is finite" is Δ_1^{ZF}.

Exercise 3 Fill in the details of the proof of Lemma 8.7 in Devlin. That is, show the equivalence:

$$\Phi(E, X) \iff \exists f[f : X \to \mathrm{On} \wedge (\forall x, y, \in X)(xEy \to f(x) < f(y))]$$

For one direction of this equivalence, one can use the Recursion Principle that is defined on page 11 and 12 in Devlin.