Seminar Constructible Sets

Handout session 6: Chapter II, sections 3-5

2018-03-28

Cheatsheet

Theorem 1 (Collapsing Lemma, I.7.1)**.** *Let X be an extensional set. Then there is a unique transitive set M and a unique bijection $\pi : X \to M$ such that*

$$\pi : \langle X, \in\rangle \cong \langle M, \in\rangle.$$

Moreover, if $Y \subseteq X$ is transitive, then $\pi|_Y = Id_Y$.

Lemma 2 (I.9.11)**.** *Let $\Phi(\bar{x})$ be any formula of LST and let $\phi(\bar{x})$ be its counterpart in $\mathscr{L}$. Then*

$$\mathbf{ZF} \vdash \forall u \forall \bar{x} \in u[\Phi^u(\bar{x}) \leftrightarrow \models_u \phi(\mathring{\bar{x}})]$$

Lemma 3 (I.9.15)**.** *Let $\Phi(\bar{x})$ be a Σ_0-formula of LST and let $\phi(\bar{x})$ be its counterpart in $\mathscr{L}$. Then*

$$\mathbf{ZF} \vdash \text{“For any transitive set } M, \forall \bar{x} \in M[\Phi(\bar{x}) \leftrightarrow \models_M \phi(\mathring{\bar{x}})]\text{”}$$

The Axiom of Choice in L

Proposition 4. *Let $x \in L_{\alpha+1}$. Then there is a formula $\phi(\vec{v})$ of $\mathscr{L}$ such that:*

$$x = \{z \in L_\alpha | \models_{L_\alpha} \phi(\mathring{z}, \mathring{L}_{\gamma_1}, \ldots, \mathring{L}_{\gamma_n})\}$$

for some ordinals $\gamma_1, \ldots, \gamma_n$. In particular this means that x can be defined by a formula which has no individual constant symbols, next to the L_{γ_i}.

With these definitions, we can define a well-order $<_L$ on sets of the constructible universe.

Definition 5 ($<_L$: a well-order of constructible sets)**.** Let $x, y \in L$. We say that $x <_L y$ if and only if either of the following conditions hold:

1. The minimal α such that $x \in L_{\alpha+1}$ is smaller than the least β such that $y \in L_{\beta+1}$.
2. The α and β defined above are the same, and the $\lessdot$-least formula $\phi(\vec{v})$ of $\mathscr{L}$ such that $x = \{z \in L_\alpha | \models_{L_\alpha} \phi(\mathring{z}, \mathring{L}_{\gamma_1}, \ldots, \mathring{L}_{\gamma_n})\}$, for some sequence of ordinals $\gamma_1, \ldots, \gamma_n$, $\lessdot$-precedes the $\lessdot$-least formula $\psi(\vec{v})$ of $\mathscr{L}$ such that $y = \{z \in L_\alpha | \models_{L_\alpha} \psi(\mathring{z}, \mathring{L}_{\gamma'_1}, \ldots, \mathring{L}_{\gamma'_n})\}$, for some sequence of ordinals $\gamma'_1, \ldots, \gamma'_n$.

3. The formulas ϕ and ψ defined above are the same, but the $<^*$-least sequence $\gamma_1, \ldots, \gamma_n$ which defines x as in condition 2 $<^*$-precedes the $<^*$-least sequence $\gamma'_1, \ldots, \gamma'_n$ which defines y as in condition 2.

Definition 6. In this definition, we construct several logical formulas, and we will combine them step by step to construct a formula which expresses a well-order.

- We define the formula $N(\alpha, x, \phi, t)$ to be an LST-formula which says that ϕ is a formula of $\mathscr{L}$, t is a finite sequence of ordinals bounded by α, ϕ has free variables $v_0, \ldots, v_n$, where n is the length of t and we have that $x = \{z \in L_\alpha | \models_{L_\alpha} \phi(\mathring{z}, \mathring{L}_{t(0)}, \ldots, \mathring{L}_{t(n-1)})\}$. An example of this is on page 73 of [1].

- Define $M(\alpha, x, \phi)$ as:
$$\exists t(N(\alpha, x, \phi, t)) \wedge \forall \phi'(\exists t'(N(\alpha, x, \phi, t')) \to (\phi = \phi' \vee \phi \lessdot \phi'))$$

- Define $P(\alpha, x, \phi, t)$ as:
$$N(\alpha, x, \phi, t) \wedge \forall t'(N(\alpha, x, \phi, t') \to (t = t' \vee t <^* t'))$$

- Define $Q(x, y, \alpha)$ as:
$$x \in L_{\alpha+1} \wedge x \notin L_\alpha \wedge y \in L_{\alpha+1} \wedge y \notin L_\alpha \wedge$$
$$(\exists \phi, \psi(M(\alpha, x, \phi) \wedge M(\alpha, x, \psi) \wedge \phi \lessdot \psi) \vee$$
$$\exists \phi(M(\alpha, x, \phi) \wedge M(\alpha, y, \phi) \wedge \exists s, t(P(\alpha, x, \phi, s) \wedge P(\alpha, y, \phi, t) \wedge s <^* t)))$$

- Now, we define the formula $\mathrm{WO}(x, y)$ as follows:
$$\exists \alpha(x \in L_\alpha \wedge y \notin L_\alpha) \vee \exists \alpha \exists w(w = L_{\max(\omega, \alpha+4)} \wedge R(x, y, \alpha, w))$$
Here we have that $R(x, y, \alpha, w)$ is $Q(x, y, \alpha)$ with all unbounded quantifiers bounded by the value w.

Lemma 7 (3.2)**.** *The formula* $\mathrm{WO}(x, y)$ *as constructed above is* $\Delta_1^{\mathbf{KP}+(V=L)}$.

Lemma 8 (3.3)**.** *Let* $\mathrm{wo}(x, y)$ *be the equivalent in* $\mathscr{L}$ *of* $\mathrm{WO}(x, y)$*. For* $x, y \in L_\alpha$ *we then have that:*
$$\mathrm{WO}(x, y) \leftrightarrow \models_{L_\gamma} \mathrm{wo}(\mathring{x}, \mathring{y})$$
where $\gamma = \max(\omega, \alpha + 5)$

Proposition 9 (3.6)**.** *There is a* Σ_1 *formula of LST* $\mathrm{Enum}(\alpha, x)$*, which is absolute for* L *and for which it holds that:*
$$\mathbf{KP} \vdash F = \{(x, \alpha) | \mathrm{Enum}(\alpha, x)\} \to F : \mathbf{On} \leftrightarrow L$$

Corollary 10 (3.8)**.** $\mathbf{ZF} \vdash (\mathrm{AC})^L$

Corollary 11 (4.1)**.** *If* **ZF** *is consistent, then so too is* **ZFC***.*

Corollary 12 (4.2)**.** *If* **ZF** *is consistent, then so too is* $\mathbf{ZFC} + (V = L)$*.*

The Generalized Continuum Hypothesis in L

Definition 13. Let $\mathbf{M}$ and $\mathbf{N}$ be structures, we say that

- $\mathbf{N}$ is a *substructure* of $\mathbf{M}$ if $N \subseteq M$ and

$$\models_{\mathbf{N}} \phi \iff \models_{\mathbf{M}} \phi$$

  for all atomic $\mathscr{L}_{\mathbf{N}}$-sentences ϕ.

- $\mathbf{N}$ is a Σ_n-*elementary substructure* of $\mathbf{M}$ (denote $\mathbf{N} \prec_n \mathbf{M}$) if the above holds for all Σ_n $\mathscr{L}_{\mathbf{N}}$-sentences.

- $\mathbf{N}$ is an *elementary substructure* of $\mathbf{M}$ (denote $\mathbf{N} \prec \mathbf{M}$) if the above holds for all $\mathscr{L}_{\mathbf{N}}$-sentences.

Theorem 14 (Condensation Lemma, 5.2)**.** *Let α be a limit ordinal. If*

$$X \prec_1 L_\alpha$$

then there are unique π and β such that $\beta \leq \alpha$ and:

(i) $\pi : \langle X, \in \rangle \cong \langle L_\beta, \in \rangle$,

(ii) for transitive $Y \subseteq X$, $\pi|_Y = Id_Y$,

(iii) $\pi(x) \leq_L x$ for all $x \in X$.

Lemma 15 (5.3)**.** *Let α be a limit ordinal, and $X \subseteq L_\alpha$. Let M be the set of all elements of L_α that are definable in L_α from X (i.e. $a \in M$ if and only if there is an $\mathscr{L}_X$-formula ϕ such that a is unique with $\models_{L_\alpha} \phi(\mathring{a})$).*

Then

$$X \subseteq M \prec L_\alpha$$

and M is the smallest such substructure.

Corollary 16 (5.4)**.** $|M| = \max(|X|, \omega)$.

Lemma 17 (5.5)**.** *Assume $V = L$. Let κ be a cardinal, and let $x \subseteq L_\alpha$ for some $\alpha < \kappa$. Then $x \in L_\kappa$.*

Theorem 18 (5.6)**.** *$V = L$ implies* GCH.

Corollary 19 (5.8)**.** *If* **ZF** *is consistent, then so too is* **ZF** + GCH.

Exercises

Exercise 1. In the proof of Lemma 8, (Lemma 3.3(i) in [1]), we give γ the value $\max(\omega, \alpha + 5)$. Show why this value works for this proof.

Exercise 2. Show that the formula $\text{Enum}(\alpha, x)$ as is shown in Lemma 3.6 in [1] fulfils the prerequisites of that lemma. That is, show that it is absolute for L and argue why the main statement holds in **KP**.

Exercise 3. In this exercise we assume $V = L$. For each of the following statements, determine whether or not they are true (and explain why).

(i) A set X is finite[1] if and only if every injection $X \to X$ is also a surjection.

(ii) There is infinite κ such that $\kappa^\kappa \neq \kappa^+$.

(iii) The first uncountable cardinal ω_1 is singular.

References

[1] Keith J. Devlin, *Constructibility*, Springer-Verlag Berlin, ISBN 0-387-13258-9, 1984.

[1] Recall that we defined a set X to be finite if there is a bijection $n \to X$ for some natural number n.