Seminar Constructible Sets

Handout session 9: Kurepa trees and Inaccessible cardinals

2018-04-18

Inaccessible cardinals

Definition 1. Let κ be an uncountable cardinal. We say that κ is *weakly inaccessible* if it is a regular limit cardinal.

If κ is a regular *strong* limit cardinal, that is, if it satisfies

$$2^\lambda < \kappa \quad \text{(for all } \lambda < \kappa\text{)},$$

then we call κ *strongly inaccessible* or just *inaccessible*.

Remark. If GCH is true, then any cardinal is weakly inaccessible if and only if it is strongly inaccessible.

Proposition 2. *In* **ZFC** *we have that* V_κ *is a model of* **ZFC** *if* κ *is inaccessible. In* **ZF** *we have that* L_κ *is a model of* **ZFC** *if* κ *is weakly inaccessible.*

Let **I** be the sentence expressing "There exists an inaccessible cardinal" and let **WI** be the sentence expressing "There exists a weakly inaccessible cardinal".

Corollary 3. *(i)* $\mathbf{ZFC} \nvdash \mathbf{I}$ *and* $\mathbf{ZFC} \nvdash \mathbf{WI}$

(ii) If **ZFC** *is consistent, then* $\mathbf{ZFC} + \neg \boldsymbol{I}$ *is consistent.*

Relative Constructibility

For this whole section, let A be an arbitrary set.

Definition 4. Given a set X, we define $Def^A(X)$ as the set of all subsets of X definable in $\langle X, \in, A \cap X \rangle$ by an $\mathscr{L}_X(\mathring{A})$-formula having one free variable. Then we define by recursion *the hierarchy of sets constructible relative to* A as

$L_0[A] = \emptyset$,

$L_{\alpha+1}[A] = Def^A(L_\alpha[A])$,

$L_\lambda[A] = \bigcup_{\alpha<\lambda} L_\alpha[A]$ for λ limit ordinal; and

the universe of sets constructible relative to A as $L[A] = \bigcup_{\alpha \in \mathbf{On}} L_\alpha[A]$.

Remark. $L \subseteq L[A]$

Lemma 5. *(i)* $\gamma \leq \alpha \implies L_\gamma[A] \subseteq L_\alpha[A]$.

(ii) $L_\alpha[A]$ *is transitive for all* α*, and thus* $L[A]$ *is transitive.*

Proposition 6. $L[A]$ *is an inner model of* **ZFC***.*

Lemma 7. *Let* $\bar{A} = A \cap L[A]$*.*

(i) $\bar{A} \in L[A]$.

(ii) $(V = L[\bar{A}])^{L[A]}$.

Kurepa Trees and Inaccessible Cardinals

Definition 8. A *Kurepa tree* is an ω_1-tree with ω_2 or more ω_1-branches.

Theorem 9 (Solovay, appears as §4 in [2])**.** *If* $X \subseteq \omega_1$ *and* $V = L[X]$*, then a Kurepa Tree exists.*

Lemma 10. *For every set* X *and any ordinal* α *we have that* $\omega_\alpha^{L[X]} \leq \omega_\alpha$ *as ordinals in* V*.*

Theorem 11 (Page 9 in [2])**.** *If there is no Kurepa Tree, then* ω_2 *is inaccessible in* L*.*

Corollary 12. *If* **ZFC** *+ "there is no Kurepa Tree" is consistent, then so too is* **ZFC** *+* ***I****.*

Exercises

Exercise 1. For an arbitrary set A, let $\bar{A} = A \cap L[A]$ and prove that for all ordinals α, $L_\alpha[A] = L_\alpha[\bar{A}]$ and thus $L[A] = L[\bar{A}]$.

Exercise 2. We will prove Lemma 10 in detail. Let X be any set, prove the following facts. In each part you may of course use the preceding parts.

(a) Show that if κ is a cardinal in V, then κ is also a cardinal in $L[X]$.

(b) Show that for any ordinal α, we have that $\omega_\alpha^{L[X]}$ is an ordinal in V.

(c) Show that for any ordinal α we have $\omega_\alpha^{L[X]} \leq \omega_\alpha$ as ordinals in V.

References

[1] Keith J. Devlin, *Constructibility*, Springer-Verlag Berlin, ISBN 0-387-13258-9, 1984.

[2] Thomas J. Jech, *Trees*, The Journal of Symbolic Logic, Vol. 36, No. 1, pp.1-14 1971.