Seminar Constructible Sets: Model Solution for Homework 10

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Lemma 1 (Homework)**.** *Given an uncountable regular cardinal λ, a set $E \subseteq \lambda$ is stationary in λ iff every reductive function $E \to \lambda$ is constant on some unbounded subset of E.*

Proof. Let λ be an uncountable regular cardinal and $E \subseteq \lambda$ a subset. By Fodor's theoreom, if $f : E \to \lambda$ is regressive then there is some $\beta \in \lambda$ such that

$$\{\alpha \in E \mid f(\alpha) = \beta\}$$

is stationary, and thus unbounded. This gives one direction of the proof.

For the other direction, suppose $E \subseteq \lambda$ is not stationary. There exists then a set C, club in λ such that $E \cap C = \emptyset$. Let f be the function $E \to \lambda$ that sends α to $\sup(C \cap \alpha)$.

We prove that f is reductive. Since every element of $C \cap \alpha$ is bounded by α, we certainly have $f(\alpha) \leq \alpha$. Suppose now that $f(\alpha) = \alpha$. Then α is a limit point of C, and thus (by closedness) lies in C. However, we assumed $E \cap C = \emptyset$ and $\alpha \in E$, a contradiction.

It remains to show that f is not constant on any unbounded subset of E. Suppose, on the contrary, that such a set $A \subset E$ exists. Pick $\alpha, \alpha' \in A$ and $\gamma \in C$ such that $\alpha < \gamma < \alpha'$. By assumption, $f(\alpha') = f(\alpha) < \alpha$, but at the same time $f(\alpha') = \sup(C \cap \alpha') \geq \gamma$, giving a contradiction. $\square$

The arguments detailed in the first and fourth paragraph are worth 3 points each, while those in the second and third paragraph are worth 2 points each.