Seminar Constructible Sets

Model solution session 3

2018-03-14

Exercise 1

In **ZF** we 'only' have the axiom of "set foundation", that is:

$$\forall x(x \neq \emptyset \to \exists y \in x(x \cap y = \emptyset)).$$

In **BS** we have the axiom of "full foundation", which may seem stronger. Prove that "full foundation" is a derivable in **ZF** (hint: you will want to use the transitive closure of a set described in [1, page 12]).

Solution to exercise 1 (4pt)

Let A be a non-empty class, we will show that there is a set in A that is disjoint from A. Let x be a set in A, if it happens to be the case that $x \cap A = \emptyset$, then we are done already. (1pt)

If $x \cap A$ is non-empty, we consider $\mathrm{TC}(x)$, the transitive closure of x, and note that since $x \subseteq \mathrm{TC}(x)$ we have that $\mathrm{TC}(x) \cap A \neq \emptyset$. (1pt)

By separation, $\mathrm{TC}(x) \cap A$ is a set, so using set foundation we find $y \in \mathrm{TC}(x) \cap A$ such that $y \cap \mathrm{TC}(x) \cap A = \emptyset$. (1pt)

We claim that this y is also disjoint from A. Suppose it is not, then there is $z \in y \cap A$. Since $z \in y \in \mathrm{TC}(x)$ and $\mathrm{TC}(x)$ is transitive, we must have that $z \in \mathrm{TC}(x)$. However, that means that $z \in y \cap \mathrm{TC}(x) \cap A$, which contradicts the choice of y. Therefore y is disjoint from A, which concludes our proof. (1pt)

Exercise 2

Let T be a theory, $n \in \mathbb{N}$ and let $\phi(x)$ be a Σ_n formula such that

$$T \vdash \exists x(\phi(x))$$
$$T \vdash \phi(x) \leftrightarrow \forall y(\phi(y) \to x = y)$$

Show that $\phi(x)$ is Δ_n^T.

Solution to exercise 2 (2pt)

The second condition is actually sufficient. If $n = 0$ then the statement is trivial. Otherwise, since ϕ is a Σ_n formula, there is a Π_{n-1} formula $\phi'(x, \vec{y})$ such that $\phi(x)$ is of the form $(\exists \vec{z})\phi'(x, \vec{z})$. We can thus rewrite the second condition to be

$$T \vdash \phi(x) \leftrightarrow \forall y((\exists \vec{z})\phi'(y, \vec{z}) \to x = y).$$

By contraposition, this is equivalent to

$$T \vdash \phi(x) \leftrightarrow \forall y(x \neq y \to (\forall \vec{z})\neg\phi'(x, \vec{z})).$$

Since for any α and β with $\vec{z}$ not free in α, $\alpha \to (\forall \vec{z})\beta$ is equivalent to $(\forall \vec{z})(\alpha \to \beta)$, we can move the universal quantifier and the quantifiers in $\neg\phi'$ to the front, giving a T-equivalence between $\phi(x)$ and a Π_n formula ($\neg\phi'$, being the negation of a Π_{n-1} formula, is Σ_{n-1}).

Exercise 3

An attempt at integer addition for $n, m \in \mathbb{N}$ is a function $A : \omega \times \omega \to \omega$ such that for all $n' \leq n$ and $m' \leq m$, $A(n', m') = n' + m'$.

Show that the property "A is an attempt at integer addition for n, m" can be expressed as a Δ_0 formula. (You may use lemma 8.4 from [1].)

Solution to exercise 3 (4pt)

The following formula works:

$$\begin{aligned}
\mathrm{At}(n, m, A) : \text{`}A\text{ is a function'} \\
&\wedge \mathrm{dom}(A) = \omega \times \omega \wedge \mathrm{ran}(A) \subseteq \omega \\
&\wedge A(0, 0) = 0 \\
&\wedge (\forall n' \in n)(\forall m' \in m)(A(n' + 1, m') = A(n', m') + 1) \\
&\wedge (\forall n' \in n)(\forall m' \in m)(A(n', m' + 1) = A(n', m') + 1) \\
&\wedge (0 \in n \wedge 0 \in m \to A(n, m) = A(n - 1, m) + 1).
\end{aligned}$$

We can express that A is a function by lemma 8.4. We can express $\mathrm{dom}(A) = \omega \times \omega$ by

$$\begin{aligned}
&(\forall p \in A)(p_{1,0} \in \omega \wedge p_{1,1} \in \omega) \\
\wedge &(\exists p \in A)(p_1 = (0, 0)) \\
\wedge &(\forall p \in A)(\exists q \in A)(q_{1,0} = p_{1,0} + 1) \\
\wedge &(\forall p \in A)(\exists q \in A)(q_{1,1} = p_{1,1} + 1);
\end{aligned}$$

a similar argument works for $\mathrm{ran}(A) \subseteq \omega$. $A(x, y) = z$ can be expressed by $(\exists p \in A)(p_{0,0} = x \wedge p_{0,1} = y \wedge p_1 = z)$.

Depending on the desired use, $n \in \omega$ and $m \in \omega$ may be added as conditions. The question leaves ambiguous whether the intended reading is "give a formula such that, if n and m are natural numbers, expresses ..." or "give a formula that expresses that n and m are natural numbers and ..."—since the difference is trivial, both are fine.

References

[1] Keith J. Devlin, *Constructibility*, Springer-Verlag Berlin, ISBN 0-387-13258-9, 1984.