Seminar on Set Theory

Hand-out lecture 1

September 18, 2015

Part I - Lattices and algebras

Definition.

(i) We say that a poset $(L, \leq)$ is a ***bounded lattice*** if every pair of elements $x, y \in L$ has a supremum/join $x \vee_L y$ and an infimum/meet $x \wedge_L y$, and there are a greatest element 1_L and a least element 0_L.

(ii) We say that a bounded lattice is ***distributive*** is for all $x, y, z \in L$, we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

(iii) A bounded lattice is ***complete*** if every subset $A \subset L$ has a supremum $\bigvee A$ and an infimum $\bigwedge A$.

Properties. Let $(L, \leq)$ be a bounded lattice. For all $x, y, z \in L$, we have:

$$\begin{aligned} x \vee 0 &= x, & x \wedge 1 &= x, \\ x \vee y &= y \vee x, & x \wedge y &= y \wedge x, \\ x \vee (y \vee z) &= (x \vee y) \vee z, & x \wedge (y \wedge z) &= (x \wedge y) \wedge z, \\ x \vee x &= x, & x \wedge x &= x, \\ (x \vee y) \wedge y &= y, & (x \wedge y) \vee y &= y. \end{aligned}$$

Definition. A bounded lattice $(H, \leq)$ is a ***Heyting algebra*** if for all $x \in H$, the set $\{z \in H \mid z \wedge x \leq y\}$ has a greatest element. We call this element the ***implication*** of x and y, and is denoted by $x \Rightarrow_H y$. We define the ***pseudocomplement*** x^* of an $x \in H$ as $x \Rightarrow 0$.

Properties. Let $(H, \leq)$ be a Heyting algebra. For all $x, y, z \in H$, we have:

- $z \wedge x \leq y$ iff $z \leq (x \Rightarrow y)$;
- $(x \Rightarrow y) \wedge x \leq y$;
- if $y \leq z$, then $(x \Rightarrow y) \leq (x \Rightarrow z)$;
- $(x \Rightarrow y) = 1$ iff $x \leq y$;
- $(x \Leftrightarrow y) = 1$ iff $x = y$;
- $(x \Rightarrow (y \Rightarrow z)) = ((x \wedge y) \Rightarrow z)$;
- $y \leq x^*$ iff $x \wedge y = 0$ iff $x \leq y^*$;
- $x \wedge x^* = 0$;
- $x \leq x^{**}$;
- $x^* = x^{***}$;

Proposition 1. *Every Heyting algebra is distributive.*

As a result, we get another property of Heyting algebras: $(x \vee y)^* = x^* \wedge y^*$ for all $x, y \in H$.

Proposition 2. (Bell, proposition 0.1, variant.) *Given are a bounded lattice $(L, \leq)$ and an operation $\Rightarrow: L^2 \to L$. Then $\Rightarrow$ makes L into a Heyting algebra if and only if the following are satisfied:*

(i) $(x \Rightarrow x) = 1$;

(ii) $(x \Rightarrow y) \wedge x \leq y$;

(iii) $y \leq (x \Rightarrow y)$;

(iv) $(x \Rightarrow (y \wedge z)) = (x \Rightarrow y) \wedge (x \Rightarrow z)$.

Proposition 3. *Let $(L, \leq)$ a complete bounded lattice. This is a Heyting algebra if and only if $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \wedge y_i)$ for all $x, y_i \in L$. In this case, we also have $\left(\bigvee_{i\in I}\right)^* = \bigwedge_{i\in I} x_i^*$ for all $x_i \in H$.*

Definition.

(i) Let $(L, \leq)$ be a bounded lattice, and $a \in L$. A ***complement*** of a is a $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$.

(ii) A ***Boolean algebra*** is a Heyting algebra $(H, \leq)$ such that x^* is a complement of x for all $x \in H$.

Remark. The Boolean algebras are precisely the bounded lattices equipped with a complement operation.

Proposition 4. (Bell, proposition 0.2.) *Let $(H, \leq)$ be a Heyting algebra. Then $x \vee x^* = 1$ for all $x \in H$ if and only if $x^{**} = x$ for all $x \in H$.*

Properties. Let $(B, \leq)$ be a Boolean algebra. For all $x, y \in B$, we have:

- $x \vee x^* = 1$;
- $x^{**} = x$;
- $(x \wedge y)^* = x^* \vee y^*$.

If B is complete, then for all $x, x_i, y_i \in B$, we have

- $x \vee \bigwedge_{i\in I} y_i = \bigwedge_{i\in I}(x \vee y_i)$;
- $\left(\bigwedge_{i\in I} x_i\right)^* = \bigvee_{i\in I} x_i^*$.

Part II - Filters and ultrafilters

Filters and Ideals

Let L be a bounded distributive lattice. A ***filter*** F on L is a subset $F \subset L$ such that:

- $1_L \in F$ and $0_L \notin F$
- if $x, y \in F$ then $x \wedge_L y \in F$
- if $x \in F$ and $x \leq y$ then $y \in F$

An ***ideal*** I on L is a subset $I \subseteq L$ such that:

- $0_L \in I$ and $1_L \notin I$
- if $x, y \in I$ then $x \vee_L y \in I$
- if $x \in I$ and $y \leq x$ then $y \in I$

The set

$$X^+ = \{y \in L \ : \ \exists x_1, ..., x_n \in X(x_1 \wedge_L ... \wedge_L x_n \leq y)\}$$

is called the ***filter generated by*** X. A filter F is called ***prime*** if $x \in F$ or $y \in F$ whenever $x \vee_L y \in F$. A filter that is maximal under inclusion is called an ***ultrafilter***.

Proposition 0.3 *Let $F \subset L$ be filter and let $b \in L - F$. Then there is a filter F' containing F and maximal under inclusion with respect to $b \notin F'$. Any such filter is prime.*

Corollary 0.4 *For any $a, b \in L$ with $a \not\leq b$ we can find a prime filter containing a but not b.*
Corollary 0.5 *Each filter in a bounded lattice is contained in an ultrafilter.*

If L, L' are distributive lattices, then $h : L \to L'$ is a ***lattice homomorphism*** if:

- $h(0_L) = 0_{L'}$ and $h(1_L) = 1_{L'}$
- $h(x \wedge_L y) = h(x) \wedge_{L'} h(y)$ and $h(x \vee_L y) = h(x) \vee_{L'} h(y)$ for all $x, y \in L$

If L and L' are Heyting algebras and $h(x \Rightarrow y) = h(x) \Rightarrow h(y)$ for all $x, y \in L$ then h is an ***algebra homomorphism***. A bijective homomorphism is called an ***isomorphism*** and if the domain and codomain are equal it is called an ***automorphism***.
Let B be a Boolean algebra and S a family of subsets of B such that every $X \in S$ has a join $\bigvee X$. An ultrafilter U in B is S***-complete*** if for all $X \in S$ we have $\bigvee X \in U$ implies $X \cap U \neq \emptyset$.
Theorem 0.6 (Rasiowa-Sikorski) *If S is a countable family of subsets of a Boolean algebra B and every member of S has a join, then for each $a \neq 0_B$ in B there is an S-complete ultrafilter in B containing a.*