Seminar on Set Theory

Handout Lecture 15, part 1

15 Januari 2016

Axioms of set theory:

(1) Extensionality: $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$.
(2) Pairing: $\forall u \forall v \exists x \forall y(y \in x \leftrightarrow y = u \vee y = v)$.
(3)* Separation: $\forall u \exists v \forall x[x \in v \leftrightarrow x \in u \wedge \varphi(x)]$.
(4) Union: $\forall u \exists v \forall x[x \in v \leftrightarrow \exists y \in u(x \in y)]$.
(5) Power set: $\forall u \exists v \forall x[x \in v \leftrightarrow \forall y \in x(y \in u)]$.
(6) Infinity: $\exists u[\emptyset \in u \wedge \forall x \in u \exists y \in u(x \in y)]$.
(7a)* Replacement: $\forall u[\forall x \in u \exists! y \varphi(x, y) \rightarrow \exists v \forall y[y \in v \leftrightarrow \exists x \in u \varphi(x, y)]]$.
(7b)* Collection: $\forall u[\forall x \in u \exists y \varphi(x, y) \rightarrow \exists v \forall x \in u \exists y \in v \varphi(x, y)]$.
(8a) Regularity: $\forall u[u \neq \emptyset \rightarrow \exists x \in u(x \cap u = \emptyset)]$.
(8b)* Set Induction: $\forall x[\forall y \in x \varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x)$.
(9) Choice: $\forall u \exists f[\mathrm{Fun}(f) \wedge \mathrm{dom}(f) = u \wedge \forall x \in u[x \neq \emptyset \rightarrow f(x) \in x]]$.

Here * marks the axiom schemes (with conditions on freeness of variables).

What we have seen so far:
- Z = axioms (1) through (6).
- ZF = ZF + (7a) + (8a).
- ZFC = ZF + (9).

Intuitionistic set theories:
- IZ = Z (with the underlying logic being intuitionistic).
- IZF_R = IZ + (7a) + (8b).
- IZF = IZ + (7b) + (8b).

Constructive alternative CZF, a weaker version of IZF modified as follows:
- Separation is given only for restricted formulas $\varphi(x)$.
- Collection is strengthened to Strong Collection: $\forall u[\forall x \in u \exists y \varphi(x, y) \rightarrow \exists v[\forall x \in u \exists y \in v \varphi(x, y) \wedge \forall y \in v \exists x \in u \varphi(x, y)]]$.
- Power set is weakened to Subset Collection: $\forall u \forall v \exists w \forall z[\forall x \in u \exists y \in v \varphi(x, y, z) \rightarrow \exists v' \in w[\forall x \in u \exists y \in v' \varphi(x, y, z) \wedge \forall y \in v' \exists x \in u \varphi(x, y, z)]]$.