Seminar on Set Theory

Hand-out lecture 2

September 25, 2015

1 Algebras and Logic

Definition 1.1. If H, H' are Heyting algebras, an *algebra homomorphism* is a lattice homomorphism $f\colon H \to H'$ such that $f(x \Rightarrow y) = f(x) \Rightarrow f(y)$ for any $x, y \in H$.

Definition 1.2. A *subalgebra* H' of a Heyting algebra H is a sublattice of H such that $x \Rightarrow y \in H'$ for any $x, y \in H'$.

Proposition 1.3. *For Boolean algebras B and B', a map $f\colon B \to B'$ is an algebra homomorphism if and only if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x^*) = f(x)^*$ for all $x, y \in B$.*

In this case, we also have $f(0) = 0$ and $f(1) = 1$ (since any algebra homomorphism is also a lattice homomorphism).

Theorem 1.4. **(Theorem 0.8 in the book)** *Any Heyting algebra is isomorphic to a subalgebra of $O(X)$ for some topological space X.*

Proof of Stone Representation Theorem. Since Boolean algebras are complemented distributive bounded lattices (see last week's hand-out) and B is a Boolean algebra, it suffices to show that $\tilde{f}\colon B \to f(B)$ respects B's * operation. (The maps $\tilde{f}$ and $f\colon B \to \mathcal{P}(F(B))$ are the same maps as in the proof of Theorem 0.7.)

Notice:

$$\begin{aligned} f(b^*) &= \{F \in F(B) \mid b^* \in F\} \\ &= \{F \in F(B) \mid b \notin F\} \quad (\text{since } b \wedge b^* = 0 \notin F) \\ &= F(B) \setminus f(b) \\ &= f(b)^*. \end{aligned}$$

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Remark 1.5. For algebras as *semantics* of propositional or first-order logic, see page 14 (halfway) and 15 of the book.