Seminar on Set Theory

Hand-out Lecture 4

October 9, 2015

Subalgebras and Submodels

We show that
$$V^{(B)} \supset V^{(2)} \cong V$$
in a natural sense.

Definition 1. Given a *complete subalgebra* of a Boolean algebra B is a subalgebra B' such that for any $X \subset B'$, $\bigvee X$ and $\bigwedge X$ exist in B' and coincide with $\bigvee X$ and $\bigwedge X$ formed in B.

Theorem 2. Let B be a Boolean algebra and B' a complete subalgebra of B. Then $V^{(B')} \subset V^{(B)}$ and for any $u, v \in V^{(B')}$ we have
$$\llbracket u = v \rrbracket^B = \llbracket u = v \rrbracket^{B'}$$
$$\llbracket u \in v \rrbracket^B = \llbracket u \in v \rrbracket^{B'}.$$

Corollary 3. Given a Boolean algebra B and a complete subalgebra B', we have for any restricted formula $\phi(a_1, \dots, a_n)$ and any $u_1, \dots, u_n \in B'$ that
$$\llbracket \phi(u_1, \dots, u_n) \rrbracket^B = \llbracket \phi(u_1, \dots, u_n) \rrbracket^{B'}.$$

Hence, when B' is a complete subalgebra of B we say that $V^{(B')}$ is a *submodel* of $V^{(B)}$.

Note that $2 \subset B$ is a complete subalgebra of any Boolean algebra B, so $V^{(2)}$ is a submodel of every $V^{(B)}$.

Define $\hat{x}$ as follows by recursion on $\in$:
$$\hat{x} = \{\langle \hat{y}, 1 \rangle \,:\, y \in x\}$$

Definition 4. We say that $y \in V^{(B)}$ is a *standard element* if there is an $x \in V$ such that $y = \hat{x}$.

Note: this is *not* the same as $\llbracket y = \hat{x} \rrbracket^B = 1$!

Theorem 5. The following properties about standard elements hold:

(i) For $x \in V, u \in V^{(B)}$,
$$\llbracket u \in \hat{x} \rrbracket^B = \bigvee_{y \in x} \llbracket u = \hat{y} \rrbracket^B.$$

(ii) For $x, y \in V$,

$$x \in y \text{ iff } V^{(B)} \models \hat{x} \in \hat{y}$$
$$x = y \text{ iff } V^{(B)} \models \hat{x} = \hat{y}$$

(iii) The map $x \mapsto \hat{x}$ is one-to-one.

(iv) For each $u \in V^{(2)}$ there is a unique $x \in V$ such that $V^{(2)} \models u = \hat{x}$.

(v) For any formula $\phi(a_1, \ldots, a_n)$ and $x_1, \ldots, x_n \in V$,

$$V \models \phi(x_1, \ldots, x_n) \text{ iff } V^{(2)} \models \phi(\hat{x}_1, \ldots, \hat{x}_n)$$

and furthermore, if ϕ is restricted,

$$V \models \phi(x_1, \ldots, x_n) \text{ iff } V^{(B)} \models \phi(\hat{x}_1, \ldots, \hat{x}_n).$$