Seminar on Set Theory

Solution for Hand-in Lecture 4

October 9, 2015

Assignment 1. *Let B be a Boolean algebra and F be a principal, $\mathcal{P}(B)$-complete ultrafilter on B. Define*

$$\pi(a) = \{\pi(x) : x \in dom(a), a(x) \in F\}.$$

Prove that for any $a, b \in V^{(B)}$,

$$[\![a \in b]\!]^B \in F \textit{ iff } \pi(a) \in \pi(b) \tag{1}$$
$$[\![a = b]\!]^B \in F \textit{ iff } \pi(a) = \pi(b). \tag{2}$$

We will abbreviate $[\![\phi]\!]^B$ to $[\![\phi]\!]$ for the remainder of this document.

Solution 1. We prove by induction on the rank of a and b. Recall that we can well-order $Ord \times Ord$ by $\langle \alpha, \beta \rangle < \langle \alpha', \beta' \rangle$ if $\langle \max\{\alpha, \beta\}, \alpha, \beta \rangle < \langle \max\{\alpha', \beta'\}, \alpha', \beta' \rangle$ lexicographically (page 18 of the book). We proceed by induction on this order. The induction hypothesis is that for all $\langle \alpha', \beta' \rangle < \langle \alpha, \beta \rangle$ we have, for all $x, y \in V^{(B)}$ with $\text{rank}(x) = \alpha'$ and $\text{rank}(y) = \beta'$:

$$[\![x \in y]\!] \in F \text{ iff } \pi(x) \in \pi(y)$$
$$[\![x = y]\!] \in F \text{ iff } \pi(x) = \pi(y).$$

We prove that for all $a, b \in V^{(B)}$ with $\text{rank}(a) = \alpha$ $\text{rank}(b) = \beta$, statements 1 and 2 hold.

Statement 1, left to right: Suppose $[\![a \in b]\!] \in F$. Recall

$$[\![a \in b]\!] = \bigvee_{x \in \text{dom}(b)} [b(x) \wedge [\![a = x]\!]].$$

By $\mathcal{P}(B)$-completeness of F, there is a $u \in \text{dom}(b)$ such that $b(x) \wedge [\![a = u]\!] \in F$. Hence $b(u) \in F$ and $[\![a = u]\!] \in F$. By the first we have $\pi(u) \in \pi(b)$. Note now that $\text{rank}(u) < \text{rank}(b)$ and thus we can apply the induction hypothesis; as $[\![a = u]\!] \in F$, $\pi(a) = \pi(u)$. Thus $\pi(a) \in \pi(b)$.

Statement 1, right to left: Suppose $\pi(a) \in \pi(b)$. Choose a $u \in \text{dom}(b)$ such that $\pi(u) = \pi(a)$ and $b(u) \in F$ (this exists by the definition of π). As $\text{rank}(u) < \text{rank}(b)$, by induction we have that $[\![a = u]\!] \in F$. Thus, as F is a filter, $b(u) \wedge [\![a = u]\!] \in F$. Now note that

$$b(u) \wedge [\![a = u]\!] \leq \bigvee_{x \in \text{dom}(b)} [b(x) \wedge [\![a = x]\!]] = [\![a \in b]\!].$$

Hence $[\![a \in b]\!] \in F$, as F is upwards closed.

Statement 2, left to right: Suppose $[\![a = b]\!] \in F$. Recall

$$[\![a = b]\!] = \bigwedge_{x \in \mathrm{dom}(a)} [a(x) \Rightarrow [\![x \in b]\!]] \wedge \bigwedge_{y \in \mathrm{dom}(b)} [b(y) \Rightarrow [\![y \in a]\!]] \,.$$

We start by showing $\pi(a) \subset \pi(b)$, and the converse inclusion follows by symmetry. Let $u \in \mathrm{dom}(a)$ such that $a(u) \in F$ (and hence, $\pi(u) \in \pi(a)$). We show $\pi(u) \in \pi(b)$.

As $[\![a = b]\!] \in F$, we have $a(u) \Rightarrow [\![u \in b]\!] \in F$. As F is a filter, $(a(u) \Rightarrow [\![u \in b]\!]) \wedge a(u) \in F$, and thus $[\![u \in b]\!] \in F$. Now, $\mathrm{rank}(u) < \mathrm{rank}(a)$ so by the induction hypothesis we have $\pi(u) \in \pi(b)$.

Statement 2, right to left: Suppose $\pi(a) = \pi(b)$. Choose $u \in \mathrm{dom}(a)$. If $a(u) \in F$ then $\pi(u) \in \pi(a)$, hence $\pi(u) \in \pi(b)$, and hence (as $\mathrm{rank}(u) < \mathrm{rank}(a)$) we by induction have $[\![u \in b]\!] \in F$. Therefore, $a(u) \wedge [\![u \in b]\!] \in F$. If $a(u) \notin F$, then $a(u)^* \in F$ because F is an ultrafilter. Thus, as $a(u) \Rightarrow [\![u \in b]\!] = a(u)^* \vee [\![u \in b]\!]$ by the properties of a Boolean algebra, we have $a(u) \Rightarrow [\![u \in b]\!] \in F$.

By the same argument we show that for any $v \in \mathrm{dom}(b)$, $b(v) \Rightarrow [\![v \in a]\!] \in F$. Hence

$$[\![a = b]\!] = \bigwedge_{x \in \mathrm{dom}(a)} [a(x) \Rightarrow [\![x \in b]\!]] \wedge \bigwedge_{y \in \mathrm{dom}(b)} [b(y) \Rightarrow [\![y \in a]\!]]$$

is an infinite meet of elements of F and is thus in F (because F is principal). Therefore, $[\![a = b]\!] \in F$.

Assignment 2. *Let α be an ordinal and for every $\xi < \alpha$, let $b_\xi \in B$, where B is a complete Boolean algebra. For each $\xi < \alpha$, we define*

$$a_\xi = b_\xi \wedge \left(\bigvee_{\eta < \xi} b_\eta \right)^* .$$

Show that

$$\bigvee_{\xi < \alpha} a_\xi = \bigvee_{\xi < \alpha} b_\xi .$$

Solution 2. We use transfinite induction, up to α, to show that the equality

$$\bigvee_{\xi < \beta} a_\xi = \bigvee_{\xi < \beta} b_\xi$$

holds for all $\beta \leq \alpha$, so in particular for $\beta = \alpha$. Let $\beta \leq \alpha$ and suppose that the equality holds for all $\gamma < \beta$. Then for all $\gamma < \beta$,

$$\bigvee_{\eta \leq \gamma} a_\eta = a_\gamma \vee \bigvee_{\eta < \gamma} a_\eta = a_\gamma \vee \bigvee_{\eta < \gamma} b_\eta = \left[b_\gamma \wedge \left(\bigvee_{\eta < \gamma} b_\eta \right)^* \right] \vee \bigvee_{\eta < \gamma} b_\eta$$

$$= \left[b_\gamma \vee \bigvee_{\eta < \gamma} b_\eta \right] \wedge \left[\left(\bigvee_{\eta < \gamma} b_\eta \right)^* \vee \left(\bigvee_{\eta < \gamma} b_\eta \right) \right] = b_\gamma \vee \bigvee_{\eta < \gamma} b_\eta = \bigvee_{\eta \leq \gamma} b_\eta .$$

It follows that

$$\bigvee_{\xi < \beta} a_\xi = \bigvee_{\gamma < \beta} \left(\bigvee_{\eta \leq \gamma} a_\eta \right) = \bigvee_{\gamma < \beta} \left(\bigvee_{\eta \leq \gamma} b_\eta \right) = \bigvee_{\xi < \beta} b_\xi ,$$

completing the induction.