Seminar on Set Theory
Hand-in Exercise 5

16 October 2015

Solutions to Exercise 1

(i) We see that $\{[\![\phi(y)]\!] : y \in V^{(B)}\}$ is a subset of B (by Separation), hence by Replacement (using the formula $\psi(x, y) \equiv y \in V^{(B)} \wedge x = [\![\phi(y)]\!]$) we have a set $v \subseteq V^{(B)}$ such that $\bigvee_{y \in V^{(B)}} [\![\phi(y)]\!] = \bigvee_{y \in v} [\![\phi(y)]\!]$. Again by Replacement (using the formula $\chi(x, y) \equiv Ord(y) \wedge x \in V_y^{(B)}$) [and Separation] we have a set $a \subseteq ORD$ such that $\forall x \in v \exists y \in a x \in V_y^{(B)}$. By applying Union to a we obtain a set $u = \{\alpha\}$ where $\chi \leq \alpha$ for all $\chi \in a$, so that we conclude our proof by obtaining $\bigvee_{y \in V^{(B)}} [\![\phi(y)]\!] = \bigvee_{y \in v} [\![\phi(y)]\!] \leq \bigvee_{y \in V_\alpha^{(B)}} [\![\phi(y)]\!] \leq \bigvee_{y \in V^{(B)}} [\![\phi(y)]\!]$.
Points awarded: 1 point for a correct proof outline, 1 point for filling in the right details such as precise use of axioms.

(ii) With the correction that the elements x must belong to a set u (otherwise we find a counterexample in the formula $\varphi(x, y) \equiv x \in V_y^{(B)}$), we can pair these x with all the elements $[\![\phi(x, y)]\!]$ and repeat the proof procedure above. That is, we have a subset $S = \{[\![\phi(x, y)]\!] : x \in u, y \in V^{(B)}\}$ of B on which we can apply Replacement to find a set $v \subseteq V^{(B)}$ such that for all $x \in u$ and all $b \in S$ we have a $y \in v$ such that $[\![\phi(x, y)]\!] = b$. We can then continue along the same lines to find our desired α such that $\bigvee_{y \in V^{(B)}} [\![\phi(x, y)]\!] = \bigvee_{y \in V_\alpha^{(B)}} [\![\phi(x, y)]\!]$.
Points awarded: $\frac{1}{2}$ point for making plausible that we can repeat the proof method of (i); $\frac{1}{2}$ point for correctly using the essential fact that the elements x belong to a set u.

Solutions to Exercise 2

(i) In the proof of Lemma 1.38 (the power set axiom) it is established that $[\![\forall x[x \in v \leftrightarrow x \subseteq u]]\!] = 1$ where $v \in V^{(B)}$ is defined by $\mathrm{dom}(v) = B^{\mathrm{dom}(u)}$ with $v(x) = [\![x \subseteq u]\!] = [\![\forall y \in x(y \in u)]\!]$ for all $x \in \mathrm{dom}(v)$. We obtain that

$v(x) = [\![\forall y \in x(y \in u)]\!] = \bigwedge_{y \in \mathrm{dom}(x)}[x(y) \Rightarrow [\![y \in u]\!]] = \bigwedge_{y \in \mathrm{dom}(u)}[x(y) \Rightarrow \bigvee_{a \in \mathrm{dom}(u)}[\![y = a]\!]]$ since $\mathrm{dom}(x) = \mathrm{dom}(u)$ for any $x \in \mathrm{dom}(v) = B^{\mathrm{dom}(u)}$ and it is given that $u(x) = 1$ for all $a \in \mathrm{dom}(u)$. It is then easy to see that the last expression is equal to 1: either $\mathrm{dom}(u)$ is empty and it follows trivially, or we can take $a = y$ to see the implication to be $x(y) \Rightarrow 1$ and hence 1. Thus we find that $v = B^{\mathrm{dom}(u)} \times \{1\} = w$, hence we can use the mentioned result to conclude that $[\![\forall x[x \in w \leftrightarrow x \subseteq u]]\!] = 1$.
Points awarded: $\frac{1}{2}$ point for a partial proof (for instance of only one side of the implication), 1 point for a complete proof.

(ii) An example which works for any Boolean algebra B can be given by defining $u = \{(\emptyset, 0)\}$ and $x = \{(\emptyset, 1)\}$. Since clearly $x \in B^{\mathrm{dom}(u)}$, we immediately obtain $1 = w(x) \leq [\![x \in w]\!]$. On the other hand we have $[\![x \subseteq u]\!] = [\![\forall y \in x[y \in u]]\!] = \bigvee_{y \in \mathrm{dom}(x)}[x(y) \Rightarrow [\![y \in u]\!]] = x(\emptyset) \Rightarrow [u(\emptyset) \wedge [\![\emptyset = \emptyset]\!]] = 1 \Rightarrow [0 \wedge 1] = 0$. Thus $[\![x \in w \rightarrow x \subseteq u]\!] = 0$, which is enough for u to be an example where $[\![\forall x[x \in w \leftrightarrow x \subseteq u]]\!] \neq 1$.
Points awarded: $\frac{1}{2}$ point for any correct example, $\frac{1}{2}$ point for showing it to be correct.

Solutions to Exercise 3

(i) For reflexivity, suppose $y \in Y$. Since Y is a core for X we have $V^{(B)} \models y \in X$; since X is a poset in $V^{(B)}$ we have $V^{(B)} \models \forall x \in X(x \leq_X x)$. Combining the two yields $V^{(B)} \models y \leq_X y$ which is $[\![y \leq_X y]\!] = 1$, and so $y \leq_Y y$.
For transitivity, suppose $y, y', y'' \in Y$ such that $y \leq_Y y'$ and $y' \leq_Y y''$. Because Y is a core for X we have $V^{(B)} \models y \in X$, $V^{(B)} \models y' \in X$ and $V^{(B)} \models y'' \in X$; from the definitions we also have $V^{(B)} \models y \leq_X y'$ and $V^{(B)} \models y' \leq_X y''$. Since X is a poset in $V^{(B)}$ we have $V^{(B)} \models \forall x \in X \forall x' \in X \forall x'' \in X(x \leq_X x' \wedge x' \leq_X x'' \rightarrow x \leq_X x'')$, thus we can combine these to obtain $V^{(B)} \models y \leq_X y''$, and so $y \leq_Y y''$.
For antisymmetry, suppose $y, y' \in Y$ such that $y \leq_Y y'$ and $y' \leq_Y y$. Again since Y is a core for X we have $V^{(B)} \models y \in X$ and $V^{(B)} \models y' \in X$, as well as $V^{(B)} \models y \leq_X y'$ and $V^{(B)} \models y' \leq_X y$ from the definitions. By X being a poset in $V^{(B)}$ we have $V^{(B)} \models \forall x \in X \forall x' \in X(x \leq_X x' \wedge x' \leq_X x \rightarrow x = x')$, hence we obtain $V^{(B)} \models y = y'$. By the definition of a core y is the unique $y \in Y$ such that $V^{(B)} \models y = y$, from which we can conclude that $y = y'$.
Points awarded: for each property, an intelligible proof outline is worth $\frac{1}{2}$ point, with another $\frac{1}{2}$ point if it is sufficiently motivated from the definitions of X as poset, Y as core for X, and of the relation $\leq_Y$ itself.

(ii) First we will have to show that indeed $V^{(B)} \models C' \subseteq X$. If $C = \emptyset$, then the same holds for C', hence we may assume C is nonempty. Now $[\![x \in C']\!] = \bigvee_{z \in \mathrm{dom}}(C')[C'(z) \wedge [\![x = z]\!]] = \bigvee_{y \in C}[\![x = y]\!]$ by definition. Since $C \subseteq Y$ where Y is a core for X, we have $[\![y \in X]\!] = 1$ for each $y \in C$. Thus $\bigvee_{y \in C}[\![x = y]\!] = \bigvee_{y \in C}[\![x = y \wedge y \in X]\!] \leq \bigvee_{y \in C}[\![x \in X]\!] = [\![x \in X]\!]$, giving us $[\![x \in C' \rightarrow x \in X]\!] = 1$ as desired.
It remains to show that C' is totally ordered, i.e. $V^{(B)} \models \forall x \in C' \forall x' \in C'$ $(x \leq_X x' \vee x' \leq_X x)$. To this end we shall prove that $[\![x \in C' \wedge x' \in C']\!] \leq [\![x \leq_X x' \vee x' \leq_X x]\!]$. Since C is a chain, we have $y \leq_Y y' \vee y' \leq_Y y$ for all $y, y' \in C$, so $[\![y \leq_X y' \vee y' \leq_X y]\!] = 1$ for all $y, y' \in C$. After some rewriting which uses $C'(x) = C'(x') = 1$ and distributivity we find $[\![x \in C' \wedge x' \in C']\!] = \bigvee_{z \in C} \bigvee_{z' \in C}[\![x = z \wedge x' = z']\!]$. Combining the two gives $[\![x \in C' \wedge x' \in C']\!] = \bigvee_{z \in C} \bigvee_{z' \in C}[[\![x = y \wedge x' = y']\!] \wedge [\![y \leq_X y' \vee y' \leq_X y]\!]] \leq \bigvee_{z \in C} \bigvee_{z' \in C}[\![x \leq_X x' \vee x' \leq_X x]\!] = [\![x \leq_X x' \vee x' \leq_X x]\!]$ as required.
Points awarded: $\frac{1}{2}$ point for showing that $V^{(B)} \models C' \subseteq X$, $\frac{1}{2}$ point for a viable proof strategy for showing C' is a total order, 1 point for providing the necessary derivations.