Seminar on Set Theory

Solutions to exercise 8

November 27, 2015

Exercise 1.

Let us call the family in the exercise $\mathcal{F}$. Let $\mathcal{P}_{\text{fin}}I$ denote the set of finite subsets of I. For $S \in \mathcal{F}$, let $I_{S,1}$ and $I_{S,0}$ be those finite subsets of I such that $f \in 2^I$ is an element of S if and only if $f(i) = 1$ for all $i \in I_{S,1}$ and $f(i) = 0$ for all $i \in I_{S,0}$.

Define a map $f\colon \mathcal{F} \to \mathcal{P}_{\text{fin}}I \times \mathcal{P}_{\text{fin}}I$ by $S \mapsto (I_{S,1}, I_{S,0})$. This map is clearly injective, so we have $|\mathcal{F}| \leq |\mathcal{P}_{\text{fin}}I \times \mathcal{P}_{\text{fin}}I| = |\mathcal{P}_{\text{fin}}I| \times |\mathcal{P}_{\text{fin}}I| = |\mathcal{P}_{\text{fin}}I|$, by the hint. (Note that we can use the hint, because $\mathcal{P}_{\text{fin}}I$ contains all singletons, so $|\mathcal{P}_{\text{fin}}I| \geq |I|$ and I is infinite.)

Furthermore, we have a natural injection from $\mathcal{P}_{\text{fin}}I$ into $\bigcup_{n\in\omega} I^n$. Hence, $|\mathcal{P}_{\text{fin}}I| \leq |\bigcup_{n\in\omega} I^n| = \sum_{n=1}^{\infty} |I^n| = \aleph_0 \times \aleph_\alpha = \aleph_\alpha$ by the hint and induction.

Thus, $|\mathcal{F}| \leq |\mathcal{P}_{\text{fin}}I| \leq \aleph_\alpha$. But the map $g\colon I \to \mathcal{F}$ defined by $i \mapsto \{f \in 2^I \mid f(i) = 1\}$ is clearly an injection, so that $\aleph_\alpha = |I| \leq |\mathcal{F}|$.

By the Cantor–Schröder–Bernstein Theorem, $|\mathcal{F}| = \aleph_\alpha$. ■

Exercise 2.

(a) By Lemma 1.52 (with $u = \mathcal{P}\hat{\kappa} \in V^{(B)}$), we get:

$$V^{(B)} \models |\mathcal{P}\hat{\kappa}| \leq |\widehat{\operatorname{dom}(\mathcal{P}\hat{\kappa})}|.$$

Note: $|\operatorname{dom}(\mathcal{P}\hat{\kappa})| = |B^{\operatorname{dom}(\hat{\kappa})}| = |B|^{|\operatorname{dom}(\hat{\kappa})|} = |\lambda^\kappa|$, because $\operatorname{dom}(\hat{\kappa}) = \{\hat{\alpha} \mid \alpha < \kappa\}$ and because the hat-map is injective.

By 1.48, we get: $V^{(B)} \models |\widehat{\operatorname{dom}(\mathcal{P}\hat{\kappa})}| = |\widehat{\lambda^\kappa}|$, so

$$V^{(B)} \models |\mathcal{P}\hat{\kappa}| \leq |\widehat{\lambda^\kappa}|,$$

as desired. ■

(b) First of all, notice that $|\omega \times \omega_2| = \aleph_2$. By Corollary 2.11, we find using the GCH,

$$\aleph_2 \leq |B| \leq \aleph_2^{\aleph_0} = (2^{\aleph_1})^{\aleph_0} = 2^{\aleph_1 \times \aleph_0} = 2^{\aleph_1} = \aleph_2.$$

Hence, $|B| = \aleph_2$. Part (a) yields: $V^{(B)} \models |\mathcal{P}\hat{\kappa}| \leq |\widehat{\aleph_2^{\kappa}}|$. If we take $\kappa = \aleph_1$, then $V^{(B)} \models |\mathcal{P}\hat{\aleph}_1| \leq |\widehat{\aleph_2^{\aleph_1}}|$.

Since B satisfies ccc, we have $V^{(B)} \models |\mathcal{P}\hat{\aleph}_1| = |\mathcal{P}\aleph_{\hat{1}}|$, but the formula $x = 1$ is restricted, so $V^{(B)} \models |\mathcal{P}\hat{\aleph}_1| = |\mathcal{P}\aleph_1| = 2^{\aleph_1}$.

Furthermore, assuming GCH, we find: $\aleph_2^{\aleph_1} = (2^{\aleph_1})^{\aleph_1} = 2^{\aleph_1} = \aleph_2$. So by 1.48, we have $V^{(B)} \models |\widehat{\aleph_2^{\aleph_1}}| = |\hat{\aleph}_2|$. But the formula $x = 2$ is restricted and B satisfies ccc, so $V^{(B)} \models |\widehat{\aleph_2^{\aleph_1}}| = \aleph_2$.

Hence, $V^{(B)} \models 2^{\aleph_1} \leq \aleph_2$, as we wished to show. ■

(c) By the given property, $V^{(B)} \models \forall\kappa \geq \hat{\lambda}(2^{\kappa} = \kappa^+)$. In part (b), we saw that $\lambda = \aleph_2$. Thus, $V^{(B)} \models \forall\kappa \geq \aleph_2(2^{\kappa} = \kappa^+)$ (because B satisfies ccc and the formula $x = 2$ is restricted, we have $V^{(B)} \models \hat{\aleph}_2 = \aleph_2$).

In part (b) we showed that $V^{(B)} \models 2^{\aleph_1} = \aleph_2$. Hence,

$$V^{(B)} \models \forall\kappa \geq \aleph_1(2^{\aleph_1} = \kappa^+).$$

When we combine this with Theorem 2.12, we obtain $V^{(B)} \models 2^{\aleph_0} = \aleph_2$ and hence the desired result. ■

(d) Let $T' = \text{ZFC} + \text{GCH}$ and $T = \text{ZFC} + 2^{\aleph_0} = \aleph_2 + \forall\kappa \geq \aleph_1(2^{\kappa} = \kappa^+)$. We know that $Consis(\text{ZF}) \rightarrow Consis(T')$.

Moreover, T' proves that B is a complete Boolean algebra and T' proves $[\![\sigma]\!]^B = 1_B$ for every axiom σ of ZFC, as we have seen before. Furthermore, we have just shown that T' proves $[\![2^{\aleph_0} = \aleph_1]\!]^B = 1_B$ and T' proves $[\![\forall\kappa \geq \aleph_1(2^{\kappa} = \kappa^+)]\!]^B = 1_B$.

We may conclude by Theorem 1.19 that $Consis(\text{ZF}) \rightarrow Consis(\text{ZFC} + 2^{\aleph_0} = \aleph_1 + \forall\kappa \geq \aleph_1(2^{\kappa} = \kappa^+))$. ■