Seminar on Set Theory

Model solution 9

Exercise 1

Exercise 1a

(a) It suffices to show that $g(x \wedge y) = g(x) \wedge g(y)$ and $g(x^*) = g(x)^*$ for all $x, y \in B'$. So let $x, y \in B$ be arbitrary. Since h is a bijection, we have

$$g(x \wedge y) = g(x) \wedge g(y) \longleftrightarrow h(g(x \wedge y)) = h(g(x) \wedge g(y)).$$

Now $h(g(x \wedge y)) = x \wedge y$ and $h(g(x) \wedge g(y)) = h(g(x)) \wedge h(g(y)) = x \wedge y$, so indeed $g(x \wedge y) = g(x) \wedge g(y)$. Similarly

$$g(x^*) = g(x)^* \longleftrightarrow h(g(x^*)) = h(g(x)^*).$$

So since $h(g(x^*)) = x^*$ and $h(g(x)^*) = h(g(x))^* = x^*$, we conclude that $g(x^*) = g(x)^*$. Hence, g is a homomorphism. (1.5 points)

Exercise 1b

Note that π and π^{-1} are both order preserving, as they are homomorphisms. Let $X \subseteq B$ and suppose that $\bigvee X$ exists in B. Since $\bigvee X \geq x$ for all $x \in X$, we have $\pi\left(\bigvee X\right) \geq \pi(x)$ for all $x \in X$, so $\pi\left(\bigvee X\right)$ is an upper bound for $\{\pi(x) \mid x \in X\}$. Suppose that $y \in B$ is also an upper bound for $\{\pi(x) \mid x \in X\}$. Then $y \geq \pi(x)$ for all $x \in X$, so $\pi^{-1}(y) \geq x$ for all $x \in X$. It follows that $\pi^{-1}(y)$ is an upper bound for X, so $\pi^{-1}(y) \geq \bigvee X$. This implies $y \geq \pi\left(\bigvee X\right)$. We conclude that $\pi\left(\bigvee X\right)$ is the least upper bound for $\{\pi(x) \mid x \in X\}$, so $\pi\left(\bigvee X\right) = \bigvee\{\pi(x) \mid x \in X\}$. Hence, π is a complete homomorphism. (1.5 points)

Exercise 1c

Suppose that B is homogeneous and let $x \neq 0$, $y \neq 0$ be in B. If we let $\pi' \in \mathrm{Aut}(B)$, then

$$\pi'\left(\bigvee\{\pi(x) \mid \pi \in \mathrm{Aut}(B)\}\right) = \bigvee\{\pi'(\pi(x)) \mid \pi \in \mathrm{Aut}(B)\},$$

by part **b**. Furthermore

$$\bigvee\{\pi'(\pi(x)) \mid \pi \in \mathrm{Aut}(B)\} = \bigvee\{\pi(x) \mid \pi \in \mathrm{Aut}(B)\},$$

since $\pi'\pi$ runs through $\mathrm{Aut}(B)$ as π runs through $\mathrm{Aut}(B)$. This means that $\bigvee\{\pi(x) \mid \pi \in \mathrm{Aut}(B)\}$ is invariant, so it must have value 0 or 1, by homogeneity of B. Since $\bigvee\{\pi(x) \mid \pi \in \mathrm{Aut}(B)\} \geq \mathrm{id}(x) = x$ and $x \neq 0$, it follows that $\bigvee\{\pi(x) \mid \pi \in \mathrm{Aut}(B)\} = 1$. Hence

$$y = y \wedge \bigvee\{\pi(x) \mid \pi \in \mathrm{Aut}(B)\} = \bigvee\{y \wedge \pi(x) \mid \pi \in \mathrm{Aut}(B)\}.$$

So since $y \neq 0$, there must be $\pi \in \mathrm{Aut}(B)$ such that $y \wedge \pi(x) \neq 0$.
Conversely, suppose that B is not homogeneous. Then there exists an invariant element $y \in B$, with $y \neq 0$ and $y \neq 1$. But then $y^* \neq 0$, so if we take $x = y^*$, then we have found nonzero $x, y \in B$ such that

$$x \wedge \pi(y) = y^* \wedge \pi(y) = y^* \wedge y = 0,$$

for all $\pi \in \mathrm{Aut}(B)$. (2 points)

Exercise 2

Exercise 2a

This is shown by proving $V_\alpha^{(\Gamma)} \subseteq V_\alpha^{(B)}$ for any ordinal α. For α an ordinal, we can show $V_\alpha^{(\Gamma)} \subseteq V_\alpha^{(B)}$ by induction. Assume for all $\beta < \alpha$ we know $V_\beta^{(\Gamma)} \subseteq V_\beta^{(B)}$. Now let $x \in V_\beta^{(\Gamma)}$, then $\text{Fun}(x) \wedge \text{ran}(x) \subseteq B \wedge \exists \beta < \alpha \text{dom}(x) \subseteq V_\beta^{(\Gamma)}$, so by the induction hyopthesis $\text{Fun}(x) \wedge \text{ran}(x) \subseteq B \wedge \exists \beta < \alpha \text{dom}(x) \subseteq V_\beta^{(B)}$, so $x \in V_\alpha^{(B)}$. So by induction $V_\alpha^{(\Gamma)} \subseteq V_\alpha^{(B)}$ for any ordinal α, so $V^{(B)} \subseteq V^{(\Gamma)}$. (.5 points for a correct answer)

Exercise 2b

Let $u = \{\langle \emptyset, r\rangle\} \in V^{(B)}$. Now for $g \in \text{stab}(u)$ we know $\{\langle \emptyset, r\rangle\} = u = gu = \{\langle g\emptyset, gr\rangle\} = \{\langle \emptyset, gr\rangle\}$, i.e. $gr = r$, so $\text{stab}(u) \subseteq \text{stab}(r)$. But as $\text{stab}(r) \notin \Gamma$, $\text{stab}(u) \notin \Gamma$(because Γ is a filter of subgroups), so $u \notin V^{(\Gamma)}$. So $V^{(\Gamma)} \neq V^{(B)}$.(.5 points for a correct answer)

Exercise 2c

Note that if B' obeys $a = \bigcap_{b' \in B'} \text{stab}(b') \in \Gamma$ and it is maximal under this property, then $B' = \{b \in B | a \subseteq \text{stab}(b)\}$. For let $b \in B$ such that $a \subseteq \text{stab}(b)$. Then $\bigcap_{b' \in B'} \text{stab}(b') \cap \text{stab}(b) \in \Gamma$(because Γ is a filter of subgroups), so by maximality of B' we know $b \in B'$. So $B' \subseteq \{b \in B | a \subseteq \text{stab}(b)\}$. Now note that for any $b \in B'$ that $a = \bigcap_{b' \in B'} \text{stab}(b') \subseteq \text{stab}(b)$, so $B' = \{b \in B | a \subseteq \text{stab}(b)\}$.
Now we find that for any $x, y \in B'$ that $\text{stab}(x \wedge y) \supseteq \text{stab}(x) \cap \text{stab}(y) \supseteq a$(as if $g \in \text{stab}(x) \cap \text{stab}(y)$, then $g(x \wedge y) = gx \wedge gy = x \wedge y$), so $x \wedge y \in B'$. Similarly $x \vee y, x \Rightarrow y, x^* \in B'$, so B' is a Boolean algebra(as these operations obey the required properties, as they do in B). (note that B' is a complete Boolean algebra by for $X \subseteq B'$, $\text{stab}(\bigvee X) \supseteq \bigcap_{x \in X} \text{stab}(x) \supseteq a$ so $\bigvee X \in B'$, and similarly $\bigwedge X \in B'$). (1 point for proving that B' is a Boolean algebra)
Now we can show for any $u \in V^{(B')}$ that $\text{stab}(u) \supseteq a$ by induction on $V_\alpha^{(B')}$. As let α be an ordinal, and for any $\beta < \alpha$ we know that for any $u \in V_\beta^{(B')}$ that $\text{stab}(u) \supseteq a$. Now let $u \in V_\alpha^{(B')}$. Then $\text{Fun}(u) \wedge \text{ran}(u) \subseteq B' \wedge \exists \beta < \alpha \text{dom}(u) \subseteq V_\beta^{(B')}$. So let $g \in a$, then $gu = \{\langle gx, g(u(x))\rangle | x \in \text{dom}(u)\}$. By the induction hypothesis for any $x \in \text{dom}(u)$ we know that $gx = x$, and by definition of B' we know that for any $b \in B'$ $gb = b$, so $g(u(x)) = u(x)$. So $gu = \{\langle x, u(x)\rangle | x \in \text{dom}(u)\} = u$. So $\text{stab}(u) \supseteq a$.
Now by induction on ordinals α we can find that $V_\alpha^{(B')} \subseteq V_\alpha^{(\Gamma)}$. As let α be an ordinal, and for any $\beta < \alpha$ we know that $V_\beta^{(B')} \subseteq V_\beta^{(\Gamma)}$. Now if $x \in V_\alpha^{(B')}$ then $\text{Fun}(x) \wedge \text{ran}(x) \subseteq B' \wedge \exists \beta < \alpha \text{dom}(x) \subseteq V_\beta^{(B')}$. Then by the induction hypothesis(and $B' \subseteq B$) $\text{Fun}(x) \wedge \text{ran}(x) \subseteq B \wedge \exists \beta < \alpha \text{dom}(x) \subseteq V_\beta^{(\Gamma)}$, and by the previous part $\text{stab}(x) \supseteq a$ so $\text{stab}(x) \in \Gamma$, so $x \in V_\alpha^{(\Gamma)}$. So by induction for any ordinal α we know that $V_\alpha^{(B')} \subseteq V_\alpha^{(\Gamma)}$, so $V^{(B')} \subseteq V^{(\Gamma)}$. (1 point for correctly using induction)

Exercise 3

Exercise 3a

Let $g \in \text{stab}(u)$. Then

$$\begin{aligned}
\text{dom}(gv) &= \bigcup\{g\text{dom}(y) | y \in \text{dom}(u)\} \\
&= \bigcup\{\text{dom}(gy) | y \in \text{dom}(u)\} \\
&= \bigcup\{\text{dom}(y) | g^{-1}y \in \text{dom}(u)\} \\
&= \bigcup\{\text{dom}(y) | y \in \text{dom}(gu)\} = \text{dom}(v)
\end{aligned}$$

using the property that $\text{dom}(gy) = \{gx | x \in \text{dom}(y)\}$ for any $y \in V^{(\Gamma)}$.(.5 points)
Now for $x \in \text{dom}(gv)$ we know that:

$$\begin{aligned}
(gv)(x) &= g(v(g^{-1}(x)) \\
&= g[\![\exists y \in u[g^{-1}x \in y]]\!]^\Gamma \\
&= [\![\exists y \in gu[gg^{-1}x \in y]]\!]^\Gamma \\
&= [\![\exists y \in u[x \in y]]\!]^\Gamma = v(x)
\end{aligned}$$

So $g \in \text{stab}(v)$, so this completes the proof. (.5 points)

Exercise 3b

Let $g \in \text{stab}(u)$. Then if $x \in \text{dom}(gv)$, then $gx \in B^{\text{dom}(u)} \cap V^{(\Gamma)}$, so in other words $gx \in V^{(\Gamma)}$ and $\text{dom}(gx) = \text{dom}(u)$. But if $gx \in V^{(\Gamma)}$, then $x = g^{-1}gx \in V^{(\Gamma)}$, and $\text{dom}(g^{-1}gx) = \text{dom}(g^{-1}u) = \text{dom}(u)$. As the definition of v in this case is identical, we can just follow the proof(1 point).(if instead $v(x) = [\![x \subseteq u$ which is what was required for the proof of thm 3.19, then we just write out definitions as in 3a)