Seminar on Set Theory

Hand-in exercise 1

September 18, 2015

Note: in this exercise, you are allowed to use all properties of Heyting algebras listed on the hand-out.

Let $(H, \leq)$ be a Heyting algebra.

(a) Show that the operation $(\cdot)^* : H \to H$ is order reversing. That is, if $x \leq y$, then $y^* \leq x^*$ for all $x, y \in H$.

(b) Define $B = \{x \in H \mid x^{**} = x\}$. Show that B with the order induced from H is a Boolean algebra, with greatest element, least element, meet and complement operations induced from H, and $x \vee_B y = (x \vee_H y)^{**}$. *Remark: it suffices to show that B is a complemented bounded lattice.*

(c) Show that if H is complete, then so is B.

We call B the *regularization* of H.

Now recall that for a topological space X, the collection of opens $O(X)$ is a Heyting algebra, with operations

$$U \wedge V = U \cap V, \quad U \vee V = U \cup V, \quad U \Rightarrow V = \overset{\circ}{\widehat{(X - U) \cup V}} \text{ and } U^* = \overset{\circ}{\widehat{X - U}} .$$

We call a $U \subset X$ *regular open* if $U = \overset{\circ}{\overline{U}}$. The collection of regular opens of X is denoted by $RO(X)$.

(d) Prove that $RO(X)$ is the regularization of $O(X)$.

In particular, $RO(X)$ can be made into a complete Boolean algebra.

(e) Show that, for a Heyting algebra H, the join operation on its regularization is not necessarily induced from H itself.