Seminar on Set Theory

Hand-in exercise 13

December 18, 2015

Exercise 1 Suppose B is a Boolean algebra and $X = \{x_1, ..., x_n\} \subseteq B$ is a non-empty finite subset of B. Define for every $y \in B$ and $i \in 2$:

$$p(y,i) = \begin{cases} y & \text{if} \quad i = 1 \\ y^* & \text{if} \quad i = 0 \end{cases}$$

For every element $f \in 2^X$, let $p_f = \bigwedge_{y \in X} p(y, f(y))$. Now let $\mathcal{F} \subseteq 2^X$ be the set of all functions f such that $p_f \neq 0$. It is easily seen (so you may assume without proof) that $|\mathcal{F}| > 1$. For $A \subseteq \mathcal{F}$ we write $p_A = \bigvee_{f \in A} p_f$.

(a) *(4 points)* Prove that $C = \{p_A \mid A \subseteq \mathcal{F}\} \subseteq B$ is a Boolean subalgebra of B.

(b) *(4 points)* Let B' be the Boolean subalgebra generated by X. Prove that $B' = C$.

(c) *(2 points)* Use the relation between elements of C and elements of the form p_f to prove that a Boolean algebra generated by $n > 0$ elements contains at most 2^{2^n} elements.