Seminar on Set Theory

Hand-in exercise 15

January 15, 2016 (due Jan. 22)

Exercise 1. (*For this exercise you can use the handout as a reference*).
Prove that over the remaining axioms of ZF...

(i) Replacement and Collection are equivalent. *(1 point)*

(ii) Regularity and Set Induction are equivalent. *(2 points)*

(iii) The Axiom of Fullness is

$$\forall u \forall v \exists x [\forall y \in x \ \mathrm{TRel}(y, u, v) \wedge \forall w [\mathrm{TRel}(w, u, v) \rightarrow \exists z \in x \ z \subseteq w]].$$

Here $\mathrm{TRel}(R, u, v)$ states that R is a total relation between u and v, i.e.

$$\forall z \in R \exists x \in u \exists y \in v \ z = \langle x, y \rangle \wedge \forall x \in u \exists y \in v \langle x, y \rangle \in R.$$

Prove Subset Collection and Fullness are equivalent over the other axioms of CZF. *(2 points)*

Exercise 2. In this exercise, we work in IZF. We write 'LEM' for the Law of the Excluded Middle, stating that $\phi \vee \neg\phi$ holds for all $\mathcal{L}$-sentences ϕ. The variables α, β, γ range over ordinals.

(i) Show that $\forall \alpha, \beta \ (\alpha < \beta^+ \rightarrow \alpha \leq \beta)$ holds. *(1 point)*

(ii) Show that $\forall \alpha, \beta \ (\alpha \leq \beta \rightarrow \alpha < \beta^+)$ implies LEM. *(1 point)*

(iii) Use the previous part to show that $\forall \alpha, \beta, \gamma \ (\alpha \leq \beta < \gamma \rightarrow \alpha < \gamma)$ implies LEM as well. *(1 point)*

(iv) Let $\mathrm{WL}(\alpha)$ be the sentence expressing that α is a weak limit. Show that

$$\forall \alpha \ (\mathrm{WL}(\alpha) \vee \exists \beta \ (\alpha = \beta^+))$$

implies LEM. *(2 points)*

Hint: for parts (ii) and (iv), recall that $\{0 \mid \phi\}$ *is an ordinal.*