Seminar on Set Theory

Hand-in exercise 9

Exercise 1

(a) Let $h : B \to B'$ be a bijective homomorphism between Boolean algebras and denote its inverse by g. Show that $g : B' \to B$ is a homomorphism. (You may use the equivalent conditions mentioned on the top of page 10 in Bell.) (1.5 points.)

Recall that if B and B' are Boolean algebras, a homomorphism $h : B \to B'$ is complete if, for any $X \subseteq B$ such that $\bigvee X$ exists in B, $\bigvee\{h(x) \mid x \in X\}$ exists in B' and equals $h(\bigvee X)$.

(b) Let $\pi : B \to B$ be an automorphism of the Boolean algebra B. Show that π is a complete homomorphism. (1.5 points.)

(c) Let B be a complete Boolean algebra. Show that B is homogeneous if and only if for each $x \neq 0$, $y \neq 0$ in B there is $\pi \in \mathrm{Aut}(B)$ such that $x \wedge \pi y \neq 0$. (Consider $\bigvee\{\pi y \mid \pi \in \mathrm{Aut}(B)\}$.) (2 points)

Exercise 2

(a) Let G be a group acting on a Boolean algebra B, and let Γ be a filter of subgroups of G.
Now prove that $V^{(\Gamma)} \subseteq V^{(B)}$. (0.5 points.)

(b) Let G be a group acting on a Boolean algebra B, with a non-invariant object $r \in B$, and let Γ be a filter of subgroups of G with the property $\mathrm{stab}(r) = \{g \in G | gr = r\} \notin \Gamma$.
Now prove that $V^{(\Gamma)} \neq V^{(B)}$. (0.5 points.)

(c) Let G be a group acting on a Boolean algebra B, and let Γ be a filter of subgroups of G.
Show that if $B' \subset B$ with $\bigcap_{b \in B'} \mathrm{stab}(b) \in \Gamma$ such that B' is maximal(under inclusion) with this property, then B' is a Boolean algebra and $V^{(B')} \subseteq V^{(\Gamma)}$. (2 points.)

Exercise 3

This exercise will be about proving that $V^{(\Gamma)}$ makes the axiom of replacement and union true, by constructing elements similar as used in the proof of lemma 1.37 and 1.38, and showing that they are elements of $V^{(\Gamma)}$.
So let G be a group acting on a Boolean algebra B, and let Γ be a normal filter of subgroups of G.

(a) For $u \in V^{(\Gamma)}$, define v by $\mathrm{dom}(v) = \bigcup\{\mathrm{dom}(y) | y \in \mathrm{dom}(u)\}$ and $v(x) = [\![\exists y \in u[x \in y]]\!]^{\Gamma}$.
Now show that $v \in V^{(\Gamma)}$ by showing $\mathrm{stab}(u) \subseteq \mathrm{stab}(v)$. (1 point.)

(b) For $u \in V^{(\Gamma)}$, define v by $\mathrm{dom}(v) = B^{\mathrm{dom}(u)} \cap V^{(\Gamma)}$ and $v(x) = [\![\exists y \in u[x \in y]]\!]^{\Gamma}$.
Now show that $v \in V^{(\Gamma)}$ by showing $\mathrm{stab}(u) \subseteq \mathrm{stab}(v)$. (1 point.)