Category Theory and Topos Theory, Spring 2016 Hand-In Exercises

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1 Exercises

Exercise 1 (To be handed in February 22) The category Rel of *relations* has as objects sets, and an arrow $X \to Y$ is a relation from X to Y, i.e. a subset of $X \times Y$. Composition of arrows is composition of relations: given $R : X \to Y$ and $S : Y \to Z$, the composite $SR : X \to Z$ is the set

$$SR \;=\; \{(x,z) \,|\, \text{for some } y \in Y,\ (x,y) \in R \text{ and } (y,z) \in S\}$$

a) Prove that Rel has both a terminal and an initial object, and give these explicitly.

b) Characterize the monomorphisms in Rel. Show in particular that every mono $R : X \to Y$ in Rel is a *total relation*: for all $x \in X$ there is a $y \in Y$ with $(x,y) \in R$.

c) Characterize the epimorphisms in Rel (Hint: you may find it useful to do part d) first).

d) Give a concrete description (up to equivalence of categories) of $\mathrm{Rel}^{\mathrm{op}}$.

Exercise 2 (To be handed in March 7) The category Set_p has as objects sets and as arrows *partial functions*: an arrow $X \to Y$ is a function f from a subset D_f of X to Y. Composition in Set_p is given as follows: for $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have

$$D_{gf} \;=\; \{x \in X \,|\, x \in D_f \text{ and } f(x) \in D_g\}$$

and $gf(x) = g(f(x))$ for $x \in D_{gf}$.

a) Prove that Set_p has finite products.

b) Prove that Set_p has equalizers.

c) Prove that Set_p has coequalizers of kernel pairs.

d) Prove that Set_p is regular.

Exercise 3 (To be handed in March 21) a) A *meet-semilattice* is a poset A which has finite limits. Concretely, it has a top element $\top$ and for $x, y \in A$, the meet (or glb) $x \wedge y$ exists. A morphism of meet-semilattices $A \to B$ is a function which preserves $\top$ and commutes with $\wedge$. We have a category **MSL** of meet-semilattices and their morphisms. Show that the forgetful functor $\mathbf{MSL} \to \mathrm{Set}$ has a left adjoint.

b) A *sup-lattice* is a poset which has least upper bounds of arbitrary subsets; a suplattice homomorphism preserves least upper bounds. We have a category **SL** of sup-lattices; prove that the forgetful functor $\mathbf{SL} \to \mathrm{Set}$ has a left adjoint.

c) The category **Frm** of frames (see section 4.5 of the lecture notes) has as objects frames and as morphisms functions which preserve arbitrary joins, top element and binary meets. Show that the forgetful functor $\mathbf{Frm} \to \mathrm{Set}$ has a left adjoint.

Exercise 4 (To be handed in April 4) Let $\mathcal{C}$ be a category, $S : \mathcal{C} \to \mathcal{C}$ a functor and (T, μ, η) a monad on $\mathcal{C}$. A *distributive law* for T over S is a natural transformation $c : TS \Rightarrow ST$ such that the following diagrams commute for each object x of $\mathcal{C}$:

$$\begin{tikzcd} & TSx \arrow[d, "c_x"] \\ Sx \arrow[ur, "\eta_{Sx}"] \arrow[r, "S(\eta_x)"'] & STx \end{tikzcd} \qquad \begin{tikzcd} T^2Sx \arrow[r, "T(c_x)"] \arrow[d, "\mu_{Sx}"'] & TSTx \arrow[r, "c_{Tx}"] & ST^2x \arrow[d, "S(\mu_x)"] \\ TSx \arrow[rr, "c_x"'] & & STx \end{tikzcd}$$

a) Prove that a distributive law for T over S gives rise to a functor $T{-}S : T{-}\mathrm{Alg} \to T{-}\mathrm{Alg}$ for which the following diagram commutes:

$$\begin{tikzcd} T{-}\mathrm{Alg} \arrow[r, "T-S"] \arrow[d, "U^T"'] & T{-}\mathrm{Alg} \arrow[d, "U^T"] \\ \mathcal{C} \arrow[r, "S"'] & \mathcal{C} \end{tikzcd}$$

b) Prove that for the functor $T{-}S$ constructed in a), the law c induces a natural transformation

$$F^T S \Rightarrow (T{-}S)F^T$$

c) Conclude that in the case that c is a natural isomorphism, the functor $T{-}S$ restricts to an endofunctor on the Kleisli category of T.

Exercise 5 (To be handed in April 18) We compare the categories $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $\mathrm{Set}^{\mathcal{C}_0}$: the latter is just the category with as objects $\mathcal{C}_0$-indexed families of sets $(X_C)_{C \in \mathcal{C}_0}$ and as arrows $\mathcal{C}_0$-indexed families of functions. There is an obvious forgetful functor $U : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}_0}$.

a) Prove that the functor U is monadic (that is, U has a left adjoint L such that $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is equivalent to the category of UL-algebras).

b) Conclude from a) that "limits are calculated point-wise" in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$

c) Prove that U also has a *right adjoint* R. For 1 extra bonus-point, you may try your hand at the question: is U also *comonadic*?

Exercise 6 (To be handed in/sent in May 11) Let X be a topological space with set of opens $\mathcal{O}(X)$. We define, for every open U of X, a set $\mathrm{Cov}(U)$ of "covering families" for U as follows: a (downwards closed) family $\mathcal{W}$ of open subsets of U is in $\mathrm{Cov}(U)$ if and only if there is a *finite* subset $\{W_1, \ldots, W_n\}$ of $\mathcal{W}$ such that $\bigcup_{i=1}^n W_i = U$. We allow $n = 0$: so $\emptyset \in \mathrm{Cov}(\emptyset)$.

a) Prove that Cov is a Grothendieck topology on $\mathcal{O}(X)$. We write $\mathrm{Sh}(\mathcal{O}(X), \mathrm{Cov})$ for the category of sheaves for this topology.

b) Let $\mathrm{Shv}(X)$ be the category of 'normal' sheaves on X, i.e. with respect to the topology where $\mathcal{W}$ covers U iff $\bigcup \mathcal{W} = U$.

Show that $\mathrm{Shv}(X) \subset \mathrm{Sh}(\mathcal{O}(X), \mathrm{Cov})$ and that the inclusion is, in general, strict: give an example of a space X and a presheaf F on X which is a sheaf for Cov but not a normal sheaf on X.

c) Show that the inclusion $\mathrm{Shv}(X) \to \mathrm{Sh}(\mathcal{O}(X), \mathrm{Cov})$ has a left adjoint which preserves finite limits.

2 Solutions

Solution to Exercise 1
a) (2 points) The empty set is initial in Rel since for any set X there is exactly one relation from $\emptyset$ to X: the empty relation. By the same token, $\emptyset$ is also terminal in Rel.
b) (3 points) Relations from X to Y correspond bijectively to union-preserving functions $\mathcal{P}(X) \to \mathcal{P}(Y)$ (where $\mathcal{P}(X)$ denotes the power set of X): $R \subseteq X \times Y$ corresponds to the function $\hat{R}$ defined by

$$\hat{R}(A) \;=\; \{y \in Y \mid \exists x \in A (x, y) \in R\}$$

Note that if $R \subseteq X \times Y$ corresponds to $\hat{R}$ and $S \subseteq Y \times Z$ to $\hat{S}$, then SR corresponds to the composite $\hat{S}\hat{R}$. Therefore, R is mono if and only if for any set W and $S, T \subseteq W \times X$, $\hat{R}\hat{S} = \hat{R}\hat{T}$ implies $S = T$. If we take $Z = \{*\}$, we see that this means that $\hat{R}$ is injective.
The following reformulation was worth 1 bonus point: R is mono if and only if there is a function $f : X \to Y$ with the property that for all $x \in X$ the following hold:

i) $(x, f(x)) \in R$

ii) For no $x' \neq x$ is $(x', f(x)) \in R$

From this last condition it is obvious that if R is mono, R is a total relation. Following the hint in the exercise, we first solve part d) before we do c).
d) (3 points) For every relation R from X to Y, we have the relation $R^\circ = \{(y,x) \mid (x,y) \in R\}$, which is a relation from Y to X, so a morphism $X \to Y$ in $\mathrm{Rel}^{\mathrm{op}}$. Moreover, for relations $R : X \to Y$ and $S : Y \to Z$, we have $(SR)^\circ = R^\circ S^\circ$ and of course, for the identity relation I on X we have $I^\circ = I$. This means that we have a functor $F : \mathrm{Rel} \to \mathrm{Rel}^{\mathrm{op}}$. Then F^{op} is a functor $\mathrm{Rel}^{\mathrm{op}} \to \mathrm{Rel}$, and it is trivial to see that the two functors are each othe's inverse. So $\mathrm{Rel}^{\mathrm{op}}$ is isomorphic, and hence equivalent, to Rel.
c) (2 points) $R : X \to Y$ is epi in Rel if and only if R° is mono in Rel (by d)). In order to characterize this in terms of R, we use the condition given in the solution of part b): R is epi if and only if there is a function $f : Y \to X$ such that for every $y \in Y$ the following hold:

i) $(f(y), y) \in R$

ii) For no $y' \neq y$ is $(f(y), y') \in R$.

Solution to Exercise 2 There is a straightforward solution and a slick one; let us do the straightforward solution first.
a) (3 points) The empty set is terminal in Set_p since for any set X the unique function f with $D_f = \emptyset$, is the unique partial function $X \to \emptyset$.

For sets X and Y, consider the disjoint sum $X*Y = (X\times Y)+X+Y$ together with projections $p_X : X*Y \to X$ and $p_Y : X*Y \to Y$ with $D_{p_X} = (X\times Y)+X$, $D_{p_Y} = (X \times Y) + Y$. Given two partial functions $Z \xrightarrow{f} X$, $Z \xrightarrow{g} Y$ we have a partial function $\langle f, g\rangle : Z \to X * Y$ defined by

$$\langle f,g\rangle(z) \;=\; \begin{cases} (f(z), g(z)) \in X \times Y & \text{if } z \in D_f \cap D_g \\ f(z) \in X & \text{if } z \in D_f - D_g \\ g(z) \in Y & \text{if } z \in D_g - D_f \end{cases}$$

So $D_{\langle f,g\rangle} = D_f \cup D_g$. One can check that $\langle f, g\rangle$ is the unique partial function satisfying $p_X\langle f, g\rangle = f$ and $p_Y\langle f, g\rangle = g$. So $X * Y$, together with the maps p_X and p_Y is a product cone.

b) (2 points) Given a parallel pair of partial functions $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, a partial function $h : Z \to X$ satisfies $fh = gh$ if and only if for each $z \in D_h$ the following holds: *either* $h(z) \in D_f \cap D_g$ and $f(h(z)) = g(h(z))$, *or* $h(z) \notin D_f \cup D_g$. Hence, such h factors through the inclusion $E \subset X$ where

$$E = \{x \in D_f \cap D_g \mid f(x) = g(x)\} \cup (X - (D_f \cup D_g))$$

and therefore this inclusion is the equalizer of f and g.

c) (3 points) First we do general pullbacks, constructed out of products and equalizers. For a diagram

$$\begin{tikzcd} & W \arrow[d, "h"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

the vertex of the limiting cone is

$$P = \{(w,z) \in D_h \times D_g \mid h(w) = g(z)\} \cup (W - D_h) \times (Z - D_g) + (W - D_h) + (Z - D_g)$$

Hence, given $X \xrightarrow{f} Y$, the kernel pair of f has as vertex

$$K_f = \{(x,y) \in D_f \times D_f \mid f(x) = f(y)\} \cup (X - D_f) \times (X - D_f) + (X - D_f) + (X - D_f)$$

Now, if $g : X \to Z$ coequalizes the two projections $p_1, p_2 : K_f \to X$ then for $(x,y) \in D_f \times D_f$, if $f(x) = f(y)$ then $g(x) = g(y)$. Moreover, if $x \notin D_f$ then $g(x)$ is undefined (because there is $\bar{x} \in K_f$ such that $x = p_1(\bar{x})$ and $p_2(\bar{x})$ is undefined, or vice versa). We conclude that g factors through the image of f; uniquely since $f : D_f \to \mathrm{Im}(f)$ is surjective.
d) (2 points) We see that a partial map $f : X \to Y$ is a regular epi in Set_p if and only if the function $f : D_f \to Y$ is surjective.

If

$$\begin{tikzcd} P \arrow[r, "p_W"] \arrow[d, "p_Z"'] & W \arrow[d, "h"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback diagram as in the solution of c), then it is easy to see that if h is surjective, so is p_Z: if $z \in D_g$, pick w such that $(h(w) = g(z)$; then $z = p_Z((w,z))$. If $z \notin D_g$, then $z \in (Z - D_g)$, hence z is also in the image of p_Z. We conclude that regular epis are stable under pullback. Hence Set_p is regular.

Slick solution: prove that Set_p is equivalent to the category Set_* of *pointed sets*: an object of Set_* is a pair (X, x) with$X$ a set and $x \in X$; an arrow $(X,x) \to (Y,y)$ is a function $f : X \to Y$ satisfying $f(x) = y$.

Let $F : \mathrm{Set}_p \to \mathrm{Set}_*$ the functor which sends X to $(X \cup \{X\}, X)$ (mind you, set theory teaches us that $X \notin X$ always!), and a partial function $f : X \to Y$ to the function

$$F(f)(x) \;=\; \begin{cases} f(x) & \text{if } x \in D_f \\ Y & \text{if } x \in X - D_f \\ Y & \text{if } x = X \end{cases}$$

In the converse direction we have $G : \mathrm{Set}_* \to \mathrm{Set}_p$ which sends (X,x) to $X - \{x\}$; for $f : (X,x) \to (Y,y)$ we have $G(f) : (X - \{x\}) \to (Y - \{y\})$ with $D_{G(f)} = \{x' \in X - \{x\} \mid f(x) \neq y\}$. I leave the proof that this is an equivalence to you, as well as the (pretty easy) proof that Set_* is regular.

Solution to exercise 3.

a) (3 points) A meet-semilattice $(A, \top, \wedge)$ can be seen as a monoid with the properties:

i) the monoid operation $\wedge$ is commutative

ii) the operation is idempotent: $a \wedge a = a$

This means that given a set X and a function $f : X \to A$ where $(A, \top, \wedge)$ is a meet-semilattice, its mate (under the adjunction $\mathrm{Mon} \underset{U}{\overset{(\cdot)^*}{\leftrightarrows}} \mathrm{Set}$) $\tilde{f} : X^* \to A$ will send two sequences of elements of X to the same element of A if these sequences enumerate the same finite subset of X. We see that the left adjoint to $U : \mathbf{MSL} \to \mathrm{Set}$ sends a set X to the set $\mathcal{P}_{\mathrm{fin}}(X)$ of finite subsets of X, ordered by *reverse* inclusion. The meet operation is *union*, the top element is the empty set. For a function $f : X \to A$, $\tilde{f} : \mathcal{P}_{\mathrm{fin}}(X) \to A$ sends U to $f(x_1) \wedge \cdots \wedge f(x_n)$ if $U = \{x_1, \ldots, x_n\}$; $\tilde{f}(\emptyset) = \top$.

b) (3 points) The free sup-lattice on a set X is the powerset $\mathcal{P}(X)$ of X. Given a function $f : X \to U$ where U is the underlying set of a sup-lattice, its mate is the sup-lattice homomorphism $\tilde{f} : \mathcal{P}(X) \to U$ defined by

$$\tilde{f}(A) \;=\; \bigvee \{f(a) \mid a \in A\}$$

c) (4 points) Every frame is a meet-semilattice (and every frame homomorphism is a meet-semilattice morphism), so there is an inclusion functor $\mathbf{Frm} \to \mathbf{MSL}$ and the forgetful functor $\mathbf{Frm} \to \mathrm{Set}$ factors through this inclusion. Since adjunctions compose, it is sufficient to calculate the left adjoint to the inclusion $\mathbf{Frm} \to \mathbf{MSL}$.

For any meet-semilattice A we have a frame $\mathcal{D}A$ consisting of all downwards closed subsets of A:

$$\mathcal{D}A = \{U \subseteq A \mid \forall x, y \in A (x \in U, y \leq x \Rightarrow y \in U)\}$$

There is a meet-semilattice morphism $A \xrightarrow{\eta} \mathcal{D}A$ sending $a \in A$ to $\downarrow\! a = \{x \in A \mid x \leq a\}$. By definition of meets, we have $\downarrow\!(a \wedge b) = \downarrow\! a \cap \downarrow\! b$ (and meets in $\mathcal{D}A$ are given by intersection). And $\downarrow\!\top = A$, which is the top element of $\mathcal{D}A$. Moreover, for any $\mathbf{MSL}$-morphism $f : A \to X$ where X is a frame, there is a unique factorization

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "f"'] & \mathcal{D}A \arrow[d, "\tilde{f}"] \\ & X \end{tikzcd}$$

by the frame homomorphism $\tilde{f} : \mathcal{D}A \to X$ which sends a downwards closed subset U of A to $\bigvee\{f(a) \mid a \in U\}$. So we have a functor $\mathcal{D} : \mathbf{MSL} \to \mathbf{Frm}$ which is left adjoint to the inclusion functor.

Calculating the composition of the two left adjoints Set $\to$ **MSL** $\to$ **Frm**, we find the functor F : Set $\to$ **Frm** which sends a set X to the collection of upwards closed sets of finite subsets of X, that is: $U \in F(X)$ if and only if $U \subseteq \mathcal{P}_{\text{fin}}(X)$ and for $A, B \in \mathcal{P}_{\text{fin}}(X)$, if $A \in U$ and $A \subseteq B$, then $B \in U$. And $F(U)$ is ordered by inclusion.
(1 bonuspoint) The functor F can also be described as follows. Given a set X, the *Scott topology* on $\mathcal{P}(X)$ has as opens those subsets U of $\mathcal{P}(X)$ which satisfy:

i) whenever $A \subseteq B \subseteq X$ and $A \in U$, also $B \in U$

ii) whenever $A \in U$, there is a finite subset A' of A which also belongs to U.

Then $F(X)$ is isomorphic to the frame of open subsets of $\mathcal{P}(X)$ for the Scott topology.

Solution to Exercise 4.

a) (5 points) Define $(T{-}S)$ on objects by: $(T{-}S)(Tx \xrightarrow{h} x$ is the composition of

$$TSx \xrightarrow{c_x} STx \xrightarrow{S(h)} Sx$$

and on maps $f : (Tx \xrightarrow{h} x) \to (Ty \xrightarrow{k} y)$ by

$$\begin{tikzcd}
TSx \arrow[r, "TSf"] \arrow[d, "c_x"'] & TSy \arrow[d, "c_y"] \\
STx \arrow[r, "STf"] \arrow[d, "Sh"'] & STy \arrow[d, "Sk"] \\
Sx \arrow[r, "Sf"] & Sy
\end{tikzcd}$$

It is clear that the diagram defining $(T{-}S)(f)$ commutes, that the definition is functorial, and that the diagram given in the exercise commutes; so it only remains to check that $(T{-}S)(Tx \xrightarrow{h} x)$ is a T-algebra when h is.

Now the unit diagram

$$\begin{tikzcd}
 & TSx \arrow[d, "c_x"] \\
 & STx \arrow[d, "Sh"] \\
Sx \arrow[uur, "\eta_{Sx}"] \arrow[r, "\mathrm{id}"'] & Sx
\end{tikzcd}$$

commutes since

$$\begin{aligned} S(h)c_x\eta_{Sx} &= S(h)S(\eta_x) \\ &= S(h\eta_x) \\ &= S(\mathrm{id}) = \mathrm{id} \end{aligned}$$

The first equality by naturality of c, and the second by the assumption that h is an algebra.

The associativity diagram

$$\begin{tikzcd}
T^2Sx \arrow[r, "Tc_x"] \arrow[dd, "\mu_{Sx}"'] & TSTx \arrow[r, "TS(h)"] \arrow[d, "c_{Tx}"] & TSx \arrow[d, "c_x"] \\
 & ST^2x \arrow[r, "ST(h)"'] \arrow[d, "S(\mu)"] & STx \arrow[d, "S(h)"] \\
TSx \arrow[r, "c_x"'] & STx \arrow[r, "S(h)"'] & Sx
\end{tikzcd}$$

commutes: the left-hand square because of the second diagram defining a distributive law in a); the right-hand upper square by naturality of c, and the lower right-hand square because h is an algebra.

b) (3 points) We have that $(T{-}S)(F^T(x))$ is the composition

$$TSTx \xrightarrow{c_{Tx}} ST^2x \xrightarrow{S(\mu)} STx$$

and the second diagram defining a distributive law just says that there is an algebra map

$$\begin{tikzcd}
T^2Sx \arrow[r, "T(c_x)"] \arrow[dd, "\mu_{Sx}"'] & TSTx \arrow[d, "c_{Tx}"] \\
 & ST^2x \arrow[d, "S(\mu)"] \\
TSx \arrow[r, "c_x"'] & STx
\end{tikzcd}$$

from $F^T(Sx)$ to $(T{-}S)(F^Tx)$, which is obviously natural.

c) (2 points) The exercise was sloppily formulated. What one can conclude from b) is that, when we apply the functor $T{-}S$ to a free algebra, we get something that is isomorphic to a free algebra. Therefore, the functor $T{-}S$ restricts to the full subcategory of $T-\mathrm{Alg}$ on those algebras that are isomorphic to a free algebra. This category is *equivalent* to the Kleisli category for T, but not (necessarily) isomorphic to it. If you had spotted this glitch and got stuck, you received at least 1 point.

Solution to Exercise 5.

a) (4 points) We denote a typical object of $\mathrm{Set}^{\mathcal{C}_0}$ by $X = (X_C)_C$ ($C \in \mathcal{C}_0$), and typical presheaves on $\mathcal{C}$ by F, G.

So, $U(F) = (F(C))_C$.

For $X = (X_C)_C$, let $L(X)$ be defined by

$$L(X)(C) \;=\; \{(f,\xi) \,|\, \mathrm{dom}(f) = C, \xi \in X_{\mathrm{cod}(f)}\}$$

Then $L(X)$ is a presheaf: for $g : C' \to C$ and $(f,\xi) \in L(X)(C)$, we let

$$L(X)(g)(f,\xi) \;=\; (fg,\xi)$$

Moreover, the assignment $X \mapsto L(X)$ is functorial: given an arrow $\alpha = (\alpha_C : X_C \to Y_C)$ from X to Y in $\mathrm{Set}^{\mathcal{C}_0}$, we have $L(\alpha) : L(X) \to L(Y)$ defined by

$$L(\alpha)_C(f,\xi) \;=\; (f, \alpha_{\mathrm{cod}(f)}(\xi))$$

This defines L as a functor.

There is a natural transformation $\varepsilon : LU \Rightarrow \mathrm{id}$: given a presheaf F, the component $\varepsilon^F_C : LUF(C) \to F(C)$ sends (f,ξ) to $F(f)(\xi)$. This is a natural transformation: for $\alpha : F \to G$ we have

$$\begin{array}{rcll} \varepsilon^G_C(LU(\alpha)_C(f,\xi)) & = & \varepsilon^G_C(f,\alpha_{\mathrm{cod}(f)}(\xi)) & = \\ G(f)(\alpha_{\mathrm{cod}(f)}(\xi)) & = & \alpha_C(F(f)(\xi)) & = \\ \alpha_C(\varepsilon^F_C(f,\xi)) & & & \end{array}$$

so the diagram

$$\begin{tikzcd} LU(F) \arrow[r, "LU(\alpha)"] \arrow[d, "\varepsilon^F"'] & LU(G) \arrow[d, "\varepsilon^G"] \\ F \arrow[r, "\alpha"'] & G \end{tikzcd}$$

commutes.

There is a natural transformation $\eta : \mathrm{id} \Rightarrow UL$: given X, we have $\eta_C : X_C \to UL(X)_C$ defined by

$$\eta_C(\xi) \;=\; (\mathrm{id}_C, \xi) \in LU(X)_C$$

The naturality of η is obvious. We check the triangle equalities:

$$\begin{tikzcd} U \arrow[rr, Rightarrow, "\eta\star U"] \arrow[dr, Rightarrow, "\mathrm{id}"'] & & ULU \arrow[dl, Rightarrow, "U\circ\varepsilon"] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} L \arrow[rr, Rightarrow, "L\circ\eta"] \arrow[dr, Rightarrow, "\mathrm{id}"'] & & LUL \arrow[dl, Rightarrow, "\varepsilon\star L"] \\ & L & \end{tikzcd}$$

For a presheaf F and $\xi \in (UF)_C = F(C)$, we have $(\eta \star U)_C(\xi) = (\mathrm{id}_C, \xi)$ and $(U\circ\varepsilon)_C(\mathrm{id}_C,\xi) = F(\mathrm{id}_C)(\xi) = \xi$, so the left-hand triangle commutes.

For X in $\mathrm{Set}^{\mathcal{C}_0)}$ and $(f,\xi) \in L(X)(C)$, $(L \star \eta)(f,\xi) = (\mathrm{id}_C,(f,\xi))$ and

$$\begin{array}{rcll} (\varepsilon \star L)_C(\mathrm{id}_C,(f,\xi)) & = & L(X)(\mathrm{id}_C)(f,\xi) & = \\ (f\mathrm{id}_C,\xi) & = & (f,\xi) & \end{array}$$

so also the right-hand triangle commutes. We conclude that indeed, L is left adjoint to U.

Consider the monad $T = UL$ on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$: $T(X)_C = \{(f, \xi) \mid \mathrm{dom}(f) = C, \xi \in X_{\mathrm{cod}(f)}\}$, so

$$T^2(X)_C \;=\; \{(g, (f, \xi)) \mid \mathrm{dom}(g) = C, \mathrm{cod}(g) = \mathrm{dom}(f), \xi \in X_{\mathrm{cod}(f)}\}$$

The multiplication $\mu : T^2 \Rightarrow T$ is given by $(g, (f, \xi)) = (fg, \xi)$, and the unit η is the unit of the adjunction $L \dashv U$.

Now suppose X is a T-algebra: we have $h : T(X) \to X$ such that the diagrams

$$\begin{tikzcd} T^2X \arrow[r, "\mu"] \arrow[d, "Th"'] & TX \arrow[d, "h"] \\ TX \arrow[r, "h"'] & X \end{tikzcd} \qquad \begin{tikzcd} & TX \arrow[d, "h"] \\ X \arrow[ur, "\eta"] \arrow[r, "\mathrm{id}"'] & X \end{tikzcd}$$

commute.

These diagrams mean that $h_C(f, \xi) \in X_C$ for all C, and for $C \xrightarrow{g} C' \xrightarrow{f} D$ and $\xi \in X_D$ we have the equalities

$$h_C(g, h_{C'}(f, \xi)) \;=\; h_C(fg, \xi) h_C(\mathrm{id}_C, \xi) \;=\; \xi$$

These equations just mean that X has a presheaf structure if, for $\xi \in X_D$ and $f : C \to D$ we put $X(f)(\xi) = h_C(f, \xi)$. Conversely, every structure of a presheaf on X defines a T-algebra structure on X. We see that $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is equivalent to the category T-Alg, and that U is monadic.

b) (2 points) In $\mathrm{Set}^{\mathcal{C}_0}$, limits are pointwise (a diagram in $\mathrm{Set}^{\mathcal{C}_0}$ is just a $\mathcal{C}_0$-indexed family of diagrams in Set).

Since U is monadic, it creates limits: for any diagram $P : I \to \mathrm{Set}^{\mathcal{C}_0}$, i.e. family of diagrams $(P_C : I \to \mathrm{Set})_{C \in \mathcal{C}_0}$ with limits Λ_C, there is a unique presheaf structure on Λ_C, and this is the vertex of a limiting cone for P. So, limits are calculated pointwise.

c) (4 points) Given an object X of $\mathrm{Set}^{\mathcal{C}_0}$, define $R(X)$ as follows: $R(X)(C)$ consists of all families

$$(\xi_f)_{\mathrm{cod}(f)=C}$$

for which $\xi_f \in X_{\mathrm{dom}(f)}$, for all f with codomain C.

Then $R(X)$ is a presheaf if we define, for $g : C' \to C$,

$$R(X)(g)((\xi_f)_{\mathrm{cod}(f)=C}) \;=\; (\xi_{gh})_{\mathrm{cod}(h)=C'}$$

For $\alpha : X \to Y$ in $\mathrm{Set}^{\mathcal{C}_0}$ we have $R(\alpha) : R(X) \to R(Y)$ given by

$$R(\alpha)_C((\xi_f)_{\mathrm{cod}(f)=C}) \;=\; (\alpha_{\mathrm{dom}(f)}(\xi_f))_{\mathrm{cod}(f)=C}$$

This defines R as a functor: $\mathrm{Set}^{\mathcal{C}_0} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

We have a natural transformation $\eta : \mathrm{id} \Rightarrow RU$: for a presheaf F, define η_C on $F(C)$ by

$$\eta_C(\xi) \;=\; (F(f)(\xi))_{\mathrm{cod}(f)=C}$$

Also, we have a natural transformation $\varepsilon : UR \Rightarrow \mathrm{id}$: for an object X of $\mathrm{Set}^{\mathcal{C}_0}$, ε_C sends the family $(\xi_f)_{\mathrm{cod}(f)=C}$ to $\xi_{\mathrm{id}_C} \in X_C$.

Again, we check the triangle equalities

$$\begin{tikzcd} R \arrow[rr, Rightarrow, "\eta\star R"] \arrow[dr, Rightarrow, "\mathrm{id}"'] & & RUR \arrow[dl, Rightarrow, "R\circ\varepsilon"] \\ & R & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[rr, Rightarrow, "U\circ\eta"] \arrow[dr, Rightarrow, "\mathrm{id}"'] & & URU \arrow[dl, Rightarrow, "\varepsilon\star U"] \\ & U & \end{tikzcd}$$

For the left-hand triangle, if $\xi = (\xi_f)_{\mathrm{cod}(f)=C}$ is an element of $R(X)(C)$ then $(\eta \star R)_C(\xi)$ is the family of families $(A_f)_{\mathrm{cod}(f)=C}$, where, for $f : C' \to C$,

$$A_f = (\xi_{fg})_{\mathrm{cod}(g)=C'}$$

And

$$\begin{array}{rcl} (R\circ\varepsilon)_C((A_f)_{\mathrm{cod}(f)=C}) & = & (\varepsilon_{C'}(A_f))_{f:C'\to C} \quad = \\ (\xi_f)_{f:C'\to C} & = & \xi \end{array}$$

so the left-hand triangle commutes. Virtually the same calculation shows that the right-hand triangle commutes; we conclude that R is right adjoint to U.

For the last statement (1 bonus point), we look at the comonad $\perp\!\!\!\perp \, = UR$ on $\mathrm{Set}^{\mathcal{C}_0}$. We have the *counit* $\varepsilon : \perp\!\!\!\perp \Rightarrow \mathrm{id}$, which is the counit of the adjunction $U \dashv R$, and a *comultiplication* $\delta : \perp\!\!\!\perp \Rightarrow \perp\!\!\!\perp^2$, which is $U\circ(\eta \star R)$.

A $\perp\!\!\!\perp$-coalgebra in $\mathrm{Set}^{\mathcal{C}_0}$ is an object X together with a morphism $h : X \to \perp\!\!\!\perp X$, making the diagrams

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "h"'] & \perp\!\!\!\perp X \arrow[d, "\perp\!\!\!\perp h"] \\ \perp\!\!\!\perp X \arrow[r, "\delta"'] & \perp\!\!\!\perp^2 X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "h"'] \arrow[dr, "\mathrm{id}"] & \\ \perp\!\!\!\perp X \arrow[r, "\varepsilon"'] & X \end{tikzcd}$$

commute.

So we have a map h such that in particular, for $\xi \in X_C$, $h_C(\xi)$ is a family $(\zeta_f)_{f:C'\to C}$ such that (by the right-hand triangle) $\zeta_{\mathrm{id}} = \xi$.

For $f : C' \to C$ let us denote $\zeta_f = (h_C(\xi))_f$ by $X(f)(\xi)$. By the above paragraph, $X(\mathrm{id}_C)(\xi) = \xi$. So $h_C(\xi) = (X(f)(\xi))_{\mathrm{cod}(f)=C}$ whence

$$(\perp\!\!\!\perp h)_C(h_C(\xi)) \;=\; ((X(g)(X(f)(\xi)))_{g:C''\to C'})_{f:C'\to C}$$

and

$$\delta_C(h_C(\xi)) \;=\; (X(fg)(\xi)_{g:C''\to C'})_{f:C'\to C}$$

From this we see that the requirement that the left-hand diagram commute, just means that

$$X(fg)(\xi) \quad X(g)(X(f)(\xi))$$

always; that is, X is a presheaf. We conclude that U is in fact comonadic (and, hence, creates colimits; which are, thus, calculated pointwise in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$).