A topological proof of the compactness theorem

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In this short article, I'll exhibit a direct proof of the compactness theorem without making use of any deductive proof system. Moreover, I'll derive it from topological compactness of a certain topological space, which may justify the term "compactness". First, let me state the compactness theorem:

Theorem (Compactness Theorem)**.** *Let T be a theory in a language L. If every finite subset $T' \subseteq T$ is consistent, then T is consistent.*

In "Sets, Models & Proofs", the proof of the compactness theorem is postponed, because after the treatment of the completeness theorem, the compactness theorem will easily follow. However, the compactness theorem is a *semantical* result that does not rely on any (syntactical) deductive system whatsoever. It will turn out that there just happens to be a suitable deductive system for which a completeness and soundness theorem can be proven.

In [1], Bruno Poizat makes an argument that this method should be avoided:

> "(...) This unfortunate compactness theorem was brought in by the back door, and we might say that its original modesty still does it wrong in logic textbooks. In my opinion it is a much more essential and primordial result (and thus also less sophisticated) than Gödel's completeness theorem , which states that we can formalize deduction in a certain arithmetic way; it is an error of method to deduce it from the latter."

Bruno Poizat's book takes a different approach towards model theory than most textbooks. It is based on a viewpoint towards model theory due to the French mathematician Roland Fraïssé. He argues that since model theory is a study of semantics, one should keep it free from syntactical notions of truth. Poizat's book is not that extreme, as it treats formulas and languages as in any other book, but only *after* introducing models and notions such as elementary equivalence that are considered more essential. His style is polemic and sometimes witty, so I can't resist to finish this introduction with another citation:

> "Whatever may be the current state of model theory, its past leaves unfortunately many residues in the opening pages of traditional textbooks, which profess to introduce this model theory by considerations that have nothing to do with the daily practice of model theorists: vague developments, fuzzy definitions, inadequate proofs, appeals to a supposedly natural intuition, reeking of the stale smell of metaphysics, that science for which the mathematician feels the most instinctive horror!"

One should keep in mind that the course "Foundations of Mathematics" is an introduction into mathematical logic rather than model theory. Also, note that proving the compactness theorem from the completeness theorem is by no means "wrong" in the mathematical sense.

The proof exhibited below is derived from Poizat's book, but all notions are very common in mathematical logic and topology. We start with a definition.

Definition. Let I be a set. A *filter* $\mathcal{F}$ on I is a subset of $\mathcal{P}(I)$ such that for all $D, E \subseteq I$ the following statements hold:

(i) $\varnothing \notin \mathcal{F}$, $I \in \mathcal{F}$

(ii) If $D, E \in \mathcal{F}$, then $D \cap E \in \mathcal{F}$

(iii) If $D \in \mathcal{F}$ and $E \supseteq D$, then $E \in \mathcal{F}$.

An *ultrafilter* $\mathcal{U}$ on I is a filter such that for all $D \subseteq I$, $D \in \mathcal{U}$ or $I \setminus D \in \mathcal{U}$.

Exercise 1. The set of filters on a set I is partially ordered by set-inlusion $\subseteq$.

(i) Prove that an ultrafilter $\mathcal{U}$ is the same thing as a maximal filter.

(ii) Prove that any filter $\mathcal{F}$ on a set I can be extended to an ultrafilter $\mathcal{U}$ on I.

Suppose M_i is a family of sets indexed by a set I. Let $\mathcal{U}$ be an ultrafilter on I. We denote by $\prod_{i\in I} M_i$ the cartesian product of the sets M_i.

Now define a relation $\sim$ on $\prod_{i\in I} M_i$ as follows:

$$(a_i)_{i\in I} \sim (b_i)_{i\in I} \Longleftrightarrow \{i \in I \mid a_i = b_i\} \in \mathcal{U}.$$

Exercise 2. Prove that $\sim$ is an equivalence relation.

We define the *ultraproduct* $\prod_{\mathcal{U}} M_i$ as the set of $\sim$-equivalence classes of $\prod_{i\in I} M_i$. One can view the filter $\mathcal{U}$ as a set of all majorities in a "parliament" with members in I. For two elements of $\prod_{i\in I} M_i$, a member i only looks at the projections in M_i to decide whether they are equal or not. Then two elements in $\prod_{i\in I} M_i$ are considered equal in the ultraproduct if a "majority" thinks that they are equal.

An ultrafilter $\mathcal{U}$ on a set I is called *principal* if there is a finite non-empty subset $I' \subseteq I$ such that

$$\mathcal{U} = \{D \subseteq I \mid D \supseteq I'\}.$$

Non-principal ultrafilters are not very interesting; using the same analogy as above, they may be viewed as a parliament in which only the votes of a certain finite number of members count (like in a totalitarian state).

Now suppose we have fixed a language L, and are given a set of L–structures M_i, indexed by a set I. Again, $\mathcal{U}$ denotes an ultraproduct on I. Let $\mathfrak{M} = \prod_{\mathcal{U}} M_i$. For convenience of notation, I'll often denote elements of $\mathfrak{M}$ by a representative in $\prod_{i\in I} M_i$. Of course it is important to check any definitions are independent of the choice of those representatives.

We define an interpretation of L in $\mathfrak{M}$ as follows:

1. For constants $c \in L$, let $c^{\mathfrak{M}}$ be (the equivalence class of) $(c^{M_i})_{i\in I}$.

2. For an n–ary relation symbol $R \in L$, define $R^{\mathfrak{M}}$ by:

$$(m_1, \ldots, m_n) \in R^{\mathfrak{M}} \Longleftrightarrow \{i \in I \mid (m_{1i}, \ldots, m_{ni}) \in R^{M_i}\} \in \mathcal{U}.$$

3. For an n–ary function symbol $f \in L$, define $f^{\mathfrak{M}}$ by:

$$f^{\mathfrak{M}}(m_1,\dots,m_n) = k \Longleftrightarrow \{i \in I \mid f^{M_i}(m_{1i},\dots,m_{ni}) = k_i\} \in \mathcal{U}$$

Exercise 3. Check that the above is an interpretation of L in $\mathfrak{M}$.

The following is a famous result in model theory by the Polish mathematician Jerzy Łoś (his last name is pronounced [ˈwɔɕ], approximately "wash"). We assume $(M_i)_{i\in I}, \mathcal{U}, \mathfrak{M}$ as defined above.

Theorem (Łoś' theorem)**.** *Let $\phi(x_1,\dots,x_n)$ be an arbitrary L–formula, and let $(a_1,\dots,a_n) \in \mathfrak{M}$. Then*

$$\mathfrak{M} \models \phi(a_1,\dots,a_n) \Longleftrightarrow \{i \in I \mid M_i \models \phi(a_{1i},\dots,a_{ni})\} \in \mathcal{U}.$$

In other words, a formula is true if and only if a "majority" of the models thinks it is true!

The proof of Łoś' theorem is by induction on the complexity of formulas. I will not include it here, but it can be found everywhere, or you could leave it for yourself as a (not so easy) exercise.

Now define the set $\mathcal{T}$ as the set of all complete theories T of L, that is; $T \in \mathcal{T}$ if and only if T is consistent and for all L–sentences ϕ, $\phi \in T$ or $\neg\phi \in T$.

We define a topology on $\mathcal{T}$ as follows: A basis is given by all sets U_ϕ, where ϕ is an L–sentence, defined by:

$$U_\phi = \{T \in \mathcal{T} \mid \phi \in T\}.$$

Since $U_\phi \cap U_\psi = U_{\phi\wedge\psi}$, this defines a basis.

Observe that $\mathcal{T} \setminus U_\phi = U_{\neg\phi}$, so the basis consists of sets that are both closed and open, these are sometimes referred to as *clopen*. Recall that a space X is called *Hausdorff* if for any $x, y \in X$, the points x and y have disjoint open neighbourhoods. A space X is called *totally disconnected* if the only connected subsets are single points and the empty set.

Exercise 4. Prove that $\mathcal{T}$ is a totally disconnected Hausdorff space.

A family of sets $\{Y_j\}_{j\in J}$ is said to have the *finite intersection property* (FIP) if for every finite subset $J' \subseteq J$,

$$\bigcap_{j\in J'} Y_j \neq \varnothing.$$

Exercise 5. Let X be a Hausdorff space. Show that X is compact if and only if every family of closed sets $\{Y_j\}_{j\in J}$ that has the FIP, has nonempty intersection:

$$\bigcap_{j\in J} Y_j \neq \varnothing.$$

Theorem (Compactness Theorem)**.** *The topological space $\mathcal{T}$ is compact.*

Proof. We use exercise 5. Assume that $\{Y_j\}_{j\in J}$ is a family of closed subsets of $\mathcal{T}$ with the FIP. We have to prove that it has non-empty intersection. Since every closed set is an arbitrary intersection of a basic clopen set U_ϕ, we can assume that $Y_j = U_{\phi_j}$ for some L–sentence ϕ_j, for every $j \in J$.

It is easy to see that we can extend this family to a filter, and therefore to an ultrafilter $\mathcal{U}$ on $\mathcal{T}$.

For every $T \in \mathcal{T}$, pick a model M_T of T. Consider the ultraproduct

$$\mathfrak{M} = \prod_{\mathcal{U}} M_T$$

with the interpretation described above. Let T' be the complete theory of $\mathfrak{M}$. Clearly $T' \in \mathcal{T}$, but then for all L–sentences ϕ,

$$\mathfrak{M} \models \phi \Longleftrightarrow M_{T'} \models \phi.$$

Therefore, we see by Łoś' theorem that:

$$U_\phi \in \mathcal{U} \Longleftrightarrow T' \in U_\phi.$$

Since we could assume that $Y_j = U_{\phi_j}$ for every $j \in J$, it follows that $T' \in \bigcap_{j \in J} Y_j$ and we are done. It follows that $\mathcal{T}$ is compact. □

Exercise 6. Using the fact that $\mathcal{T}$ is compact, prove the compactness theorem as stated in the beginning.

Note A topological space that is compact, Hausdorff and totally disconnected is called a *Stone space*. The space $\mathcal{T}$ is an example of a Stone space. It is an important result in universal algebra that Stone spaces are dual to boolean algebras, which establishes an intimate connection between topology and logic. What we have seen here is an instance of this connection.

References

[1] Bruno Poizat. *A course in model theory*. Universitext. Springer-Verlag, New York, 2000. An introduction to contemporary mathematical logic, Translated from the French by Moses Klein and revised by the author.