Gödel's Incompleteness Theorems
Hand-in Exercises and Model Solutions

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Exercise 1 (Exercise 38 of the lecture notes). Let $K : \mathbb{N} \to \mathbb{N}$, $G : \mathbb{N}^{k+1} \to \mathbb{N}$ and $H : \mathbb{N}^{k+3} \to \mathbb{N}$ be functions. Define F by:

$$\begin{array}{rcl} F(0,\vec{y},x) & = & G(\vec{y},x) \\ F(z+1,\vec{y},x) & = & H(z,F(z,\vec{y},K(x)),\vec{y},x) \end{array}$$

Suppose that G, H and K are primitive recursive.

a) (4 points) Prove directly, using the pairing function j and suitably adapting the proof of proposition 3.9: if $\forall x(K(x) \leq x)$, then F is primitive recursive.

b) (3 points) Define a new function F' by:

$$\begin{array}{rcl} F'(0,m,\vec{y},x) & = & G(\vec{y},K^m(x)) \\ F'(n+1,m,\vec{y},x) & = & H(n,F'(n,m,\vec{y},x),\vec{y},K^{m\dot{-}(n+1)}(x)) \end{array}$$

Recall that $K^{m\dot{-}(n+1)}$ means: the function K applied $m\dot{-}(n+1)$ times. Prove: if $n \leq m$ then $\forall k[F'(n,m+k,\vec{y},x) = F'(n,m,\vec{y},K^k(x))]$

c) (3 points) Prove by induction: $F(z,\vec{y},x) = F'(z,z,\vec{y},x)$ and conclude that F is primitive recursive, also without the assumption that $K(x) \leq x$.

Solution: There is more than one way to solve a), which was the most challenging part of the exercise. Define the function $\check{F}$ by:

$$\check{F}(z,\vec{y},x) \ = \ \langle F(z,\vec{y},0),\ldots,F(z,\vec{y},x)\rangle$$

Then $F(z, \vec{y}, x) = (\check{F}(z, \vec{y}, x))_x$, so if we can show that $\check{F}$ is primitive recursive, then so is F, being defined from $\check{F}$ by composition with primitive recursive functions.
Define an auxiliary function L by

$$L(z, u, \vec{y}, x) = \langle H(z, (u)_{K(0)}, \vec{y}, 0), \ldots, H(z, (u)_{K(x)}, \vec{y}, x)\rangle$$

Then

$$\begin{array}{rcl} L(z, u, \vec{y}, 0) & = & \langle H(z, (u)_{K(0)}, 0)\rangle \\ L(z, u, \vec{y}, x+1) & = & L(z, u, \vec{y}, x) * \langle H(z, (u)_{K(x+1)}, \vec{y}, x+1)\rangle \end{array}$$

so L is defined by primitive recursion from primitive recursive functions, hence primitive recursive.
Now for $\check{F}$ we have:

$$\begin{array}{rcl} \check{F}(0, \vec{y}, x) & = & \langle G(\vec{y}, 0), \ldots, G(\vec{y}, x)\rangle \\ \check{F}(z+1, \vec{y}, x) & = & L(z, \check{F}(z, \vec{y}, x), \vec{y}, x) \end{array}$$

(this takes a few lines of checking!) where in the first line we have a function defined by course-of-values recursion from G (so primitive recursive); and $\check{F}$ is defined by primitive recursion; so it is primitive recursive.

b) The only point here is to get the induction right. If one wishes to show $\forall n \leq m\, P(m)$ then it suffices to show: $P(0)$ and for all $n < m$, if $P(n)$ then $P(n+1)$.

For $n = 0$ we have $F'(n, m+k, \vec{y}, x) = F'(0, m+k, \vec{y}, x) = G(\vec{y}, K^{m+k}(x))$ and also

$$F'(n, m, \vec{y}, K^k(x)) = F'(0, m, \vec{y}, K^k(x)) = G(\vec{y}, K^m(K^k(x))) = G(\vec{y}, K^{m+k}(x)$$

so the statement holds for $n = 0$. Suppose $n < m$ and the statement holds for n. Since $n < m$ hence $n+1 \leq m$, we have $m+k \dot{-} (n+1) = (m \dot{-} (n+1)) + k$ (this is the point where the assumption $n < m$ is used! This does not hold in general!), so using the induction hypothesis we have:

$$\begin{array}{lc} F'(n+1, m+k, \vec{y}, x) & = \\ H(n, F'(n, m+k, \vec{y}, x), \vec{y}, K^{m+k \dot{-} (n+1)}(x)) & = \\ H(n, F'(n, m, \vec{y}, K^k(x)), \vec{y}, K^{m \dot{-} (n+1)}(K^k(x))) & = \\ F'(n+1, m, \vec{y}, K^k(x)) & \end{array}$$

This completes the induction step.

c) We have $F(0, \vec{y}, x) = G(\vec{y}, x)$ and $F'(0, 0, \vec{y}, x) = G(\vec{y}, K^0(x)) = G(\vec{y}, x)$, so for $z = 0$ the statement holds.

Suppose the statement holds for z. Since $z + 1 \dot{-} (z + 1) = 0$ we have: $F'(z{+}1, z{+}1, \vec{y}, x) = H(z, F'(z, z{+}1, \vec{y}, x), \vec{y}, x) = H(z, F'(z, z, \vec{y}, K(x)), \vec{y}, x)) = H(z, F(z, \vec{y}, K(x)), \vec{y}, x) = F(z{+}1, \vec{y}, x)$, which completes the induction step. We see that the function F is defined by composition from F' (and projection functions); hence F is primitive recursive. Since we have never used that $K(x) \leq x$ in this proof, F is primitive recursive without this assumption.

Exercise 2 Given a natural number $x > 0$ and a prime number p, by $\mathrm{ord}_p(x)$ (the order of p at x) we mean the highest number n such that p^n divides x.

a) (2 points) Give a formula $\psi(v, x)$ in $\mathcal{L}_{\mathrm{PA}}$ (but you can use the abbreviations pr, pp and $x{\restriction}v$ from the notes) which expresses that $x > 0$, v is prime and $\mathrm{ord}_v(x)$ is even.

b) (2 points) Give also such a formula $\chi(v, x)$, expressing: $x > 0$, v is prime and $\mathrm{ord}_v(x) \equiv 1$ (modulo 3).

c) (3 points) For the formula $\psi(v, x)$ from a), prove:

$$\mathrm{PA} \vdash \forall x[\forall v \leq x(\mathrm{pr}(v) \to \psi(v, x)) \to \exists y(y{\cdot}y = x)]$$

d) (3 points) Prove in PA that "the root of a non-square is irrational", that is:

$$\mathrm{PA} \vdash \forall xyz(x > 0 \wedge x{\cdot}x = y{\cdot}z{\cdot}z \to \exists v(y = v{\cdot}v))$$

Solution: a) $x > 0 \wedge \mathrm{pr}(v) \wedge \exists y(y{\cdot}y = x{\restriction}v)$
b) $x > 0 \wedge \mathrm{pr}(v) \wedge \exists y(y{\cdot}y{\cdot}y{\cdot}v = x{\restriction}v)$
c) You will not be punished for assuming without proof that for $x, y > 0$ and $\mathrm{pr}(v)$, $(xy){\restriction}v = (x{\restriction}v)(y{\restriction}v)$ but let's do this first: since $(x{\restriction}v)|x$ and $(y{\restriction}v)|y$, $(x{\restriction}v)(y{\restriction}v)|xy$ and hence, since $(x{\restriction}v)(y{\restriction}v)$ is a v-power by Exercise 56a) and by the definition of $(\cdot){\restriction}v$, $(x{\restriction}v)(y{\restriction}v)|(xy{\restriction}v)$. Conversely, if $x = (x{\restriction}v){\cdot}w$ and $y = (y{\restriction}v){\cdot}z$, then $v \not| wz$ and $xy = (x{\restriction}v)(y{\restriction}v)wz$, so $(xy){\restriction}v|(x{\restriction}v)(y{\restriction}v)$.

To prove c) we employ well-founded induction. Let $\chi(x)$ be the formula

$$\forall v \leq x(\mathrm{pr}(v) \to \psi(v, x)) \to \exists y(yy = x)$$

and assume

$$\begin{array}{ll} (1) & \forall x' < x\chi(x') \\ (2) & \forall v \leq x(\mathrm{pr}(v) \to \psi(v, x)) \end{array}$$

We have to prove that x is a square. This is trivial if $x \leq 1$ so let $x > 1$. Then x has a prime divisor v by Proposition 4.5. By assumption (2), let y satisfy $x{\restriction}v = yy$. Then $v|y$ so $vv|x$; let z satisfy $x = vvz$. We now have:

$$\begin{array}{ll} (3) & x{\restriction}v = vv(z{\restriction}v) \\ (4) & \text{for } \mathrm{pr}(w),\ w \neq v,\ x{\restriction}w = z{\restriction}w \end{array}$$

From (3) and assumption (2) we get that $z{\restriction}v$ is a square, and (4) says that if $\mathrm{pr}(w)$ and $w \neq v$ then $z{\restriction}w = x{\restriction}w$, hence also a square by assumption (2). Now $z < x$ so assumption (1) gives that z is a square, say $z = kk$. Then $x = (kv)(kv)$, hence a square, as desired.

By well-founded induction, we are done.

d) Suppose $x > 0$ and $xx = yzz$. We have to prove that y is a square, which (again) is trivial if $y \leq 1$. So, let $y > 1$ and v a prime divisor of y. We see that $(z{\restriction}v)(z{\restriction}v) \leq (x{\restriction}v)(x{\restriction}v)$ so $z{\restriction}v \leq x{\restriction}v$, so $(z{\restriction}v)|(x{\restriction}v)$. Let $x = x'(z{\restriction}v)$, $z = z'(z{\restriction}v)$. Then $x'x' = yz'z'$ so

$$y{\restriction}v = (yz'z'){\restriction}v = (x'{\restriction}v)(x'{\restriction}v)$$

so $y{\restriction}v$ is a square. The number v was an arbitrary prime divisor of y, therefore by c) we can conclude that y is a square.

Remark: the induction in c) is *necessary*: without the induction axioms, it is possible that there is a (nonstandard) model in which "$\sqrt{2}$ is rational": there are nonstandard elements p, q for which $p^2 = 2q^2$.

Exercise 3 This combines exercises 65 and 71 from the notes: give a full proof of Theorem 4.13 but now, with "Σ_1-formula" replaced by "Δ_1-formula" (in definition 4.12).

Solution. For a primitive recursive function F, let us write φ_F for the representing Σ_1-formula constructed in the proof of 4.13. To be explicit:
If F is $\lambda x.0$ then ϕ_F is $z = 0$
If F is $\lambda x.x + 1$ then ϕ_F is $z = x + 1$
If F is $\lambda x_1 \cdots x_k.x_i$ then φ_F is $z = x_i$
If $F(\vec{x}) = G(H_1(\vec{x}), \dots, H_m(\vec{x}))$ then φ_F is

$$\exists w_1 \cdots w_m(\varphi_{H_1}(\vec{x}, w_1) \wedge \cdots \wedge \varphi_{H_m}(\vec{x}, w_m) \wedge \varphi_G(\vec{w}, z))$$

If $F(\vec{x}, 0) = G(\vec{x})$ and $F(\vec{x}, y + 1) = H(\vec{x}, F(\vec{x}, y), y)$ then φ_F is

$$\exists am(\varphi_G(\vec{x}, (a, m)_0) \wedge \forall i < y \varphi_H(\vec{x}, (a, m)_i, i, (a, m)_{i+1}) \wedge (a, m)_y = z)$$

We prove the following things, all by induction on the definition of F as a primitive recursive function:

1. For all $n_1, \ldots, n_k \in \mathbb{N}$, PA $\vdash \varphi_F(\overline{n_1}, \ldots, \overline{n_k}, \overline{F(n_1, \ldots, n_k)})$
2. PA $\vdash \forall\vec{x}\exists! z\varphi_F(\vec{x}, z)$
3. The formula φ_F is, in PA, equivalent to a Π_1-formula.

For the basic functions, assertions 1–3 are immediate; note that φ_F is a Δ_0-formula in these cases.

In the case of **composition**: $F(\vec{x}) = G(H_1(\vec{x}), \ldots, H_m(\vec{x}))$ we assume 1–3 for $\varphi_G, \varphi_{H_1}, \ldots \varphi_{H_m}$.

1. Suppose $\vec{x} = x_1, \ldots, x_k$. Given $n_1, \ldots, n_k \in \mathbb{N}$ we have
PA $\vdash \varphi_{H_1}(\overline{n_1}, \ldots, \overline{n_k}, \overline{H_1(n_1, \ldots, n_k)})$,..., PA $\vdash \varphi_{H_m}(\overline{n_1}, \ldots, \overline{n_k}, \overline{H_m(n_1, \ldots, n_k)})$
and PA $\vdash \varphi_G(\overline{H_1(n_1, \ldots, n_k)}, \ldots, \overline{H_m(n_1, \ldots, n_k)}, \overline{F(n_1, \ldots, n_k)})$
so

$$\begin{aligned}\text{PA} \vdash \exists\vec{w}(\varphi_{H_1}(\overline{n_1} \ldots, \overline{n_k}, w_1) \wedge \cdots \wedge \varphi_{H_m}(\overline{n_1}, \ldots, \overline{n_k}, w_m)\\ \wedge\varphi_G(\vec{w}, \overline{F(n_1, \ldots,_k)}))\end{aligned}$$

so PA $\vdash \varphi_F(\overline{n_1}, \ldots, \overline{n_k}, \overline{F(n_1, \ldots, n_k)})$.

2. Reason in PA: given $\vec{x}$, we have $w_1, \ldots, w_m$ with $\varphi_{H_1}(\vec{x}, w_1), \ldots, \varphi_{H_m}(\vec{x}, w_m)$, by induction hypothesis on $H_1, \ldots, H_m$. By induction hypothesis on G we get a z with $\varphi_G(w_1, \ldots, w_m, z)$. So we have a z with $\varphi_F(\vec{x}, z)$.

For uniqueness, suppose $\varphi_F(\vec{x}, z) \wedge \varphi_F(\vec{x}, z')$. Ten we have $w_1, \ldots, w_m, w'_1, \ldots, w'_m$ with $\varphi_{H_1}(\vec{x}, w_1), \ldots, \varphi_{H_m}(\vec{x}, w_m)$ and $\varphi_{H_1}(\vec{x}, w'_1)$,...,$\varphi_{H_m}(\vec{x}, w'_m)$ and $\varphi_G(\vec{w}, z)$, $\varphi_G(\vec{w'}, z')$. The uniqueness in the induction hypothesis for $H_1, \ldots, H_m$ gives $w_1 = w'_1$,...,$w_m = w'_m$; the uniqueness in the induction hypothesis for G now gives $z = z'$

3. Let ψ_G be a Π_1-formula such that PA $\vdash \varphi_G(\vec{x}, z) \leftrightarrow \psi_G(\vec{x}, z)$. Define the formula $\psi'(\vec{x}, z)$ by

$$\forall\vec{w}(\varphi_{H_1}(\vec{x}, w_1) \wedge \cdots \wedge \varphi_{H_m}(\vec{x}, w_m) \to \psi_G(\vec{w}, z))$$

Since the φ_{H_i} are Σ_1, the formula ψ_G is Π_1, the logical equivalence $\forall x(\exists y A \to \forall w B) \leftrightarrow \forall xyw(A \to B)$ gives a Π_1-formula ψ_F equivalent to ψ'_F.

We prove that ψ'_F is equivalent to φ_F. Given $\vec{x}, z$, assume $\psi'_F(\vec{x}, z)$. By property 2 for $H_1, \ldots, H_m$, there are $w_1, \ldots, w_m$ with $\varphi_{H_1}(\vec{x}, w_1)$,...,$\varphi_{H_m}(\vec{x}, w_m)$. Hence by $\psi'_F(\vec{x}, z)$ we obtain $\psi_G(\vec{w}, z)$ hence $\varphi_G(\vec{x}, z)$. So we have $\varphi_F(\vec{x}, z)$.

Conversely, suppose $\varphi_F(\vec{x}, z)$ and assume $\varphi_{H_1}(\vec{x}, w_1)$,...,$\varphi_{H_m}(\vec{x}, w_m)$. By $\varphi_F(\vec{x}, z)$ we find $w'_1, \ldots, w'_m$ such that

$$\varphi_{H_1}(\vec{x}, w'_1) \wedge \cdots \wedge \varphi_{H_m}(\vec{x}, w'_m) \wedge \varphi_G(w'_1, \ldots, w'_m, z)$$

The uniqueness in the induction hypothesis for $H_1, \ldots, H_m$ gives $w_i = w'_i$ for $i = 1, \ldots, n$. So we get $\varphi_G(\vec{w}, z)$ and hence $\psi_G(\vec{w}, z)$ using the induction hypothesis on G. Hence $\psi'_F(\vec{x}, z)$ follows.

In the case of **primitive recursion**:

1. Given $n_1,\ldots,n_k,l$ we prove that $\mathrm{PA}\vdash\varphi_F(\overline{n_1},\ldots,\overline{n_k},\bar{l},\overline{F(n_1,\ldots,n_k,l)})$ by induction on l.

For $l=0$ we must prove $\mathrm{PA}\vdash\exists am(\varphi_G(\overline{n_1},\ldots,\overline{n_k},(a,m)_0)\wedge(a,m)_0=\overline{F(n_1,\ldots,n_k)})$ which follows from 4.9 i).

Inductively, suppose $\mathrm{PA}\vdash\varphi_F(\overline{n_1},\ldots,\overline{n_k},\bar{l},\overline{F(n_1,\ldots,n_k,l)})$ so there is a,m with

$$\begin{aligned}\mathrm{PA}\vdash\varphi_G(\overline{n_1},\ldots,\overline{n_k},(a,m)_0)\wedge\forall i<\bar{l}\varphi_H(\overline{n_1},\ldots,\overline{n_k},(a,m)_i,i,(a,m)_{i+1})&\\ \wedge(a,m)_{\bar{l}}=\overline{F(n_1,\ldots,n_k,l)}&\end{aligned}$$

By 4.9 ii), find b,n such that $\forall i\leq\bar{l}(a,m)_i=(b,n)_i$ and $(b,n)_{\overline{l+1}}$ is the unique w such that $\varphi_H(\overline{n_1},\ldots,\overline{n_k},(a,m)_{\bar{l}},\bar{l},w)$. Then this (b,n) tesifies that $\mathrm{PA}\vdash\varphi_F(\overline{n_1},\ldots,\overline{n_k},\overline{l+1},\overline{F(n_1,\ldots,n_k,l+1)})$.

2. In PA, let $\vec{x},y$ be given; to show $\exists z\varphi_F(\vec{x},y,z)$. Induction on y. For $y=0$ this is similar to case 1: use 4.9 i). For the induction step one uses 4.9 ii) again in a very similar way to the proof of 1.

To get uniqueness of z: suppose $\varphi_F(\vec{x},y,z)\wedge\varphi_F(\vec{x},y,z')$. Then there are a,m,b,n such that

$$\begin{aligned}\varphi_G(\vec{x},0,(a,m)_0)\wedge\forall i<y\varphi_H(\vec{x},(a,m)_i,i,(a,m)_{i+1})&\\ \wedge z=(a,m)_y&\\ \varphi_G(\vec{x},0,(b,n)_0)\wedge\forall i<y\varphi_H(\vec{x},(b,n)_i,i,(b,n)_{i+1})&\\ \wedge z'=(b,n)_y&\end{aligned}$$

One proves, using the uniqueness in the induction hypothesis for G and H, that $\forall i\leq y(a,m)_i=(b,n)_i$, hence $z=z'$.

3. Let $\psi'_F(\vec{x},y,z)$ be the formula

$$\begin{aligned}\forall am(\varphi_G(\vec{x},(a,m)_0)\wedge\forall i<y\varphi_H(\vec{x},(a,m)_i,i,(a,m)_{i+1})\to&\\ z=(a,m)_y&\end{aligned}$$

and ψ_F the obvious Π_1-equivalent of ψ'_F. Again, one employs induction on y to prove the equivalence

$$\varphi_F(\vec{x},y,z)\leftrightarrow\psi'(\vec{x},y,z)$$

In the proof, one uses the uniqueness property in the induction hypothesis, much in the way as property 3 was proved for composition.

Exercise 4. In this exercise, part h) is a **bonus exercise** and gives 1 point extra (the total of points to be gained from this exercise is therefore 11).

Recall that the following functions are primitive recursive.

The function assigning to x the Gödel number $\ulcorner \neg\chi \urcorner$ if x is the Gödel number of some formula χ. Otherwise, its value is 0.

The function assigning to (x, y, i) the Gödel number $\ulcorner \chi[t/v_i] \urcorner$ if x is the Gödel number of some term t, y is the Gödel number of some formula χ and t is free for v_i in χ. Otherwise, its value is 0.

The function assigning to a number x the Gödel number $\ulcorner \overline{x} \urcorner$.

Let Neg, Sub and Num be formulas representing these functions in `PA`, in such a way that the recursions of the latter two are provable in `PA`.

We define the sequence of theories $(T_n)_{n\in\mathbb{N}}$ by recursion: T_0 is `PA` and for $n \in \mathbb{N}$, T_{n+1} is $\mathtt{PA} + \mathrm{Con}_{T_n}$.

a) Prove that T_n is consistent for every $n \in \mathbb{N}$.
Thus, the given sequence is an ascending hierarchy of consistent theories, where each theory claims the consistency of the previous one. The goal of this exercise is to create a similar *descending* hierarchy, where each theory claims the consistency of the next one.

Now define the formula $\phi(v_0, v_1)$ as:

$$\exists a \exists b \exists c\ (\neg \exists x \overline{\mathrm{Prf}}(x, c)) \wedge \mathrm{Neg}(b, c) \wedge \mathrm{Sub}(a, v_0, 1, b) \wedge \mathrm{Num}(v_1 + 1, a).$$

b) Apply the Diagonalisation Lemma to ϕ to obtain a formula $\psi(v_1)$ and define $S_n := \mathtt{PA} + \psi(\overline{n})$. Show that, in `PA`, the formula $\psi(\overline{n})$ naturally expresses the consistency of S_{n+1}.
It may look as though we have our desired sequence. However, the S_n also need to be consistent.

c) Prove that $\mathtt{PA} \vdash \Box(\forall x\ \neg\psi(x)) \to \forall x\ \neg\psi(x)$. (Please don't explicitly formalize the argument in `PA`; just make it clear that the argument may be so formalized.)

d) Deduce that S_n is inconsistent for all $n \in \mathbb{N}$. [Hint: use Löb's Theorem]
This shows we have to be a bit more clever to solve our problem. Let $\phi'(v_0, v_1)$ be the formula

$$\neg \left[\exists a \exists b\ \overline{\mathrm{Prf}}(v_1, b) \wedge \mathrm{Neg}(a, b) \wedge \mathrm{Sub}\left(\overline{\ulcorner 0 \urcorner}, v_0, 1, a\right)\right] \to \phi(v_0, v_1).$$

e) As before, apply Lemma 5.1 to ϕ' to obtain a formula $\psi'(v_1)$ and define $S'_n := \mathtt{PA} + \psi'(\overline{n})$. Prove that S'_0 is consistent.

f) Show that, in $\mathtt{PA}$, the formula $\psi'(\overline{n})$ naturally expresses the consistency of S'_{n+1}.

g) Prove that S'_n is consistent for all $n \in \mathbb{N}$.

h) Can you explain why the argument that showed the S_n to be inconsistent doesn't work now?

Solution: whenever we use the term $\overline{\ulcorner \cdot \urcorner}$ in an $\mathcal{L}_{\mathtt{PA}}$-expression, where $\cdot$ is some expression with a Gödel number, we'll just write $\ulcorner \cdot \urcorner$.

Solution a). By induction on n. Certainly, $T_0 = \mathtt{PA}$ is consistent, since it has $\mathcal{N}$ as a model. Suppose T_n is consistent for some $n \in \mathbb{N}$. Then Con_{T_n} is true in the standard model $\mathcal{N}$. And again, all theorems of $\mathtt{PA}$ are true in $\mathcal{N}$. So $\mathcal{N} \models \mathtt{PA} + \mathrm{Con}_{T_n} = T_{n+1}$, and in particular, T_{n+1} is consistent. This concludes the induction.

Solution b). The Diagonalisation Lemma gives a formula $\psi(v_1)$ such that we have $\mathtt{PA} \vdash \forall v_1\ (\psi(v_1) \leftrightarrow \phi(\ulcorner \psi(v_1) \urcorner, v_1))$. Taking $v_1 = \overline{n}$, we see that $\mathtt{PA} \vdash \psi(\overline{n}) \leftrightarrow \phi(\ulcorner \psi(v_1) \urcorner, \overline{n})$. Notice that $\phi(\ulcorner \psi(v_1) \urcorner, \overline{n})$ is the formula

$$\exists a \exists b \exists c\ \neg \Box(c) \wedge \mathrm{Neg}(b, c) \wedge \mathrm{Sub}(a, \ulcorner \psi(v_1) \urcorner, 1, b) \wedge \mathrm{Num}(\overline{n} + 1, a). \tag{1}$$

Now $\mathtt{PA}$ proves for such a, b, c, that a is the Gödel number of $\overline{n+1}$, that b is the Gödel number of $\psi(\overline{n+1})$, and that c is the Gödel number of $\neg\psi(\overline{n+1})$. So $\mathtt{PA} \vdash \psi(\overline{n}) \leftrightarrow (\mathbf{??}) \leftrightarrow \neg\Box(\ulcorner \neg\psi(\overline{n+1}) \urcorner)$. That is, $\psi(\overline{n})$ is equivalent, in $\mathtt{PA}$, to the statement that the negation of $\psi(\overline{n+1})$ is not provable, which means exactly that S_{n+1} is consistent.

Solution c). We have

$$\mathtt{PA} \vdash \Box(\ulcorner \forall x\ \neg\psi(x) \urcorner) \to \forall x \boxdot \neg\psi(\widetilde{x}). \tag{2}$$

The above statement says that, given a proof of $\forall x\ \neg\psi(x)$, we can find a proof of the sentence that results when we substitute the x-th numeral in $\neg\psi$. Notice that $\mathtt{PA}$ proves that for all x, the x-th numeral is a closed term, and therefore always freely substitutable. So we can show in $\mathtt{PA}$ that we can transform a proof of $\forall x\ \neg\psi(x)$ into the desired proof by one application of $\forall E$. Therefore, ($\mathbf{??}$) indeed holds.

Now notice that $\neg\phi(\ulcorner \psi(x) \urcorner, x)$ is nothing else than $\boxdot \neg\psi(\widetilde{x+1})$. And certainly, for every formula $A(x)$, we have $\mathtt{PA} \vdash \forall x\ A(x) \to \forall x\ A(x+1)$. So we get

$$\mathtt{PA} \vdash \forall x \boxdot \neg\psi(\widetilde{x}) \to \forall x\ \neg\phi(\ulcorner \psi(x) \urcorner, x).$$

Finally, by our construction of ψ, we have

$$\texttt{PA} \vdash \forall x\ \neg\phi(\ulcorner\psi(x)\urcorner, x) \to \forall x\ \neg\psi(x).$$

Combining these yields the desired result.

Solution d). By the previous exercise and Löb's theorem, $\texttt{PA} \vdash \forall x\ \neg\psi(x)$. So in particular, we have for all $n \in \mathbb{N}$ that $\texttt{PA} \vdash \neg\psi(\overline{n})$, i.e. S_n is inconsistent.

Solution e). Let us denote the antecedent of ϕ' by $\omega(v_0, v_1)$. Then we get a formula $\psi'(v_1)$ such that $\texttt{PA} \vdash \forall v_1\ (\psi'(v_1) \leftrightarrow (\omega(\ulcorner\psi'(v_1)\urcorner, v_1) \to \phi(\ulcorner\psi'(v_1)\urcorner, v_1)))$. Notice that $\omega(\ulcorner\psi'(v_1)\urcorner, \overline{n})$ is the formula

$$\neg\left[\exists a \exists b\ \overline{\mathrm{Prf}}(\overline{n}, b) \wedge \mathrm{Neg}(a, b) \wedge \mathrm{Sub}\left(\ulcorner 0\urcorner, \ulcorner\psi'(v_1)\urcorner, 1, a\right)\right]$$

For such a and b, $\texttt{PA}$ proves that a is the Gödel number of $\psi'(0)$, and that b is the Gödel number of $\neg\psi'(0)$. So, taking $v_1 = \overline{n}$, we get

$$\texttt{PA} \vdash \psi'(\overline{n}) \leftrightarrow (\neg\overline{\mathrm{Prf}}(\overline{n}, \ulcorner\neg\psi'(0)\urcorner) \to \neg\Box(\ulcorner\neg\psi'(\overline{n+1})\urcorner)). \tag{3}$$

Suppose that S'_0 is inconsistent, i.e. $\texttt{PA} \vdash \neg\psi'(0)$. Let m be the smallest possible Gödel number of a proof of $\neg\psi'(0)$. Then we have $\texttt{PA} \vdash \overline{\mathrm{Prf}}(\overline{m}, \ulcorner\neg\psi'(0)\urcorner)$, while $\texttt{PA} \vdash \neg\overline{\mathrm{Prf}}(\overline{i}, \ulcorner\neg\psi'(0)\urcorner)$ for all $i < m$. Considering (**??**), we see that $\texttt{PA} \vdash \psi'(\overline{m})$, while $\texttt{PA} \vdash \psi'(\overline{i}) \leftrightarrow \neg\Box(\ulcorner\neg\psi'(\overline{i+1})\urcorner)$ for $i < m$. This means that $S'_m = \texttt{PA}$, while S'_i is $\texttt{PA}$ plus the statement expressing the consistency of S'_{i+1}, for $i < m$. Now by induction of j up to m, we can show that $S'_{m-j} = T_j$ for $j \leq m$. In particular, $S'_0 = T_m$ is consistent, by exercise a), contradiction.

Solution f). Since S'_0 is consistent, $\neg\psi'(0)$ cannot be proven in $\texttt{PA}$. So for all $n \in \mathbb{N}$, we have $\texttt{PA} \vdash \neg\overline{\mathrm{Prf}}(\overline{n}, \ulcorner\neg\psi'(0)\urcorner)$. Now (**??**) gives $\texttt{PA} \vdash \psi'(\overline{n}) \leftrightarrow \neg\Box(\ulcorner\neg\psi'(\overline{n+1})\urcorner))$. Now follow the same reasoning as in exercise b).

Solution g). By induction on n. We have already proven that S'_0 is consistent. Suppose that S'_{n+1} is inconsistent for some $n \in \mathbb{N}$, i.e. $\texttt{PA} \vdash \neg\psi'(\overline{n+1})$. By the rule D1, $\texttt{PA}$ proves $\Box(\ulcorner\neg\psi'(\overline{i+1})\urcorner)$, which we already know to be equivalent in $\texttt{PA}$ to $\neg\psi'(\overline{n})$. So S'_n is inconsistent as well. This concludes the induction.

Solution h). In our first attempt, we didn't only know that $\texttt{PA} \vdash \psi(\overline{n}) \leftrightarrow \neg\Box(\ulcorner\neg\psi(\overline{n+1})\urcorner))$ for each separate $n \in \mathbb{N}$, but this property was given to us uniformly as $\texttt{PA} \vdash \forall v_1\ (\psi(v_1) \leftrightarrow \phi(\ulcorner\psi(v_1)\urcorner, v_1))$. Through this statement, the formula $\psi(v_1)$ basically refers to itself, and thus we were able to get the result in exercise c). Thanks to the 'fail safe' in our second attempt, we cannot get this uniform statement. Indeed, in order to extract $\mathtt{PA} \vdash \forall v_1\ (\psi'(v_1) \leftrightarrow \phi(\ulcorner\psi'(v_1)\urcorner, v_1))$ from $\mathtt{PA} \vdash \forall v_1\ (\psi'(v_1) \leftrightarrow (\omega(\ulcorner\psi'(v_1)\urcorner, v_1) \to \phi(\ulcorner\psi'(v_1)\urcorner, v_1)))$, we'd need that $\mathtt{PA} \vdash \forall v_1\ \omega(\ulcorner\psi'(v_1)\urcorner, v_1)$. But this statement is equivalent in $\mathtt{PA}$ to $\neg\exists v_1 \overline{\mathrm{Prf}}(v_1, \ulcorner\neg\psi'(0)\urcorner)$, or even shorter: $\mathtt{PA} \vdash \neg\Box(\ulcorner\neg\psi'(0)\urcorner)$. But $\mathtt{PA}$ will never prove any statement saying that something is unprovable. Indeed, such a statement entails the consistency of $\mathtt{PA}$, which we cannot prove in $\mathtt{PA}$, by Gödel's Second Incompleteness Theorem.

Exercise 5. Let $\mathcal{M}$ be a nonstandard model of PA.

a) (2 points) Show that, in $\mathcal{M}$, there are nonstandard prime numbers.

b) (3 points) Show that, in $\mathcal{M}$, there exist nonstandard elements m such that m is divisible by every standard prime number.

c) (2 points) Let m be an element such as in b). Show that m has a nonstandard prime divisor.

d) (3 points) Show that there exist nonstandard elements m and m' such that the set $\{x \in \mathcal{M} \mid m \leq x \leq m + m'\}$ does not contain any prime number [Hint: for any natural number n, the sequence $n! + 2, n! + 3, \ldots, n! + n$ does not contain any prime number].

Solution: all these item use the fact that PA proves the infinitude of the set of primes (Exercise 55), and Overspill (Corollary 6.3) for the standard cut $\mathbb{N}$.
a) Since $\mathcal{M} \models \exists y(y > n \wedge \mathrm{prime}(y))$ for all standard n, 6.3 gives at once that also $\mathcal{M} \models \exists y(y > c \wedge \mathrm{prime}(y))$ for some nonstandard c. So in $\mathcal{M}$ there are nonstandard prime numbers.
b) The factorial function $x \mapsto x!$ is primitive recursive, hence provably total in PA, so the formula $\exists x \forall y(\mathrm{prime}(y) \wedge y < n \to y|x)$ holds in $\mathcal{M}$ for every standard n; hence also for a nonstandard number c. Any element $m \in \mathcal{M}$ for which $\mathcal{M} \models \forall y(\mathrm{prime}(y) \wedge y < c \to y|m)$ is a nonstandard number which is divisible by all standard primes.
c) Let m in $\mathcal{M}$ be divisible by all standard prime numbers. Then $\mathcal{M} \models \exists y(y > n \wedge \mathrm{prime}(y) \wedge y|m)$ for every standard number n, since there are infinitely many standard prime numbers. Hence this also holds for some nonstandard number, which implies that m is divisible by some nonstandard prime.
d) We use the hint and the remark in the solution of b) about the factorial function. We see that the formula $\exists x \forall y(x \leq y \leq x + n \to \neg\mathrm{prime}(y))$ is true

in $\mathcal{M}$ for every standard n; therefore by Overspill there is a nonstandard number $m' \in \mathcal{M}$ and an element $m \in \mathcal{M}$ such that $\mathcal{M} \models \forall y(m \leq y \leq m + m' \to \neg\text{prime}(y))$. Then m' must also be nonstandard, because for every standard element there is a standard prime above it.

Exercise 6.

a) (4 points) Let $\mathcal{M}$ be a countable model of PA. Show that every *bounded* subset of $\mathcal{M}$ is definable in parameters from $\mathcal{M}$, precisely when $\mathcal{M}$ is (isomorphic to) the standard model.

b) (6 points) Let $\mathcal{M}$ be any model of PA. Show that $\mathcal{M}$ has an elementary extension $\mathcal{M}'$ with the following property: for every subset X of $\mathcal{M}$ there is a subset Y of $\mathcal{M}'$ such that $Y \cap \mathcal{M} = X$, and Y is definable in $\mathcal{M}'$ in parameters from $\mathcal{M}'$. [Hint: use sequence coding, an appropriate extension of the language, and the Compactness Theorem]

Solution: for a), the countability assumption was in fact redundant. In the standard model, every bounded subset is finite, and hence definable. In a nonstandard model, the set of standard elements is bounded, but not definable by Lemma 6.2. If you wanted to use the countability condition (like the one who formulated this exercise) you could say: take a nonstandard element a. There are uncountably many subsets of $\mathcal{M}$ bounded by a, but there can be only countably many subsets of $\mathcal{M}$ which are definable in parameters from $\mathcal{M}$, since the language $\mathcal{L}_{\text{PA}}(\mathcal{M})$ is countable.

b) Let $\mathcal{L}'$ be the language $\mathcal{L}_{\text{PA}}(\mathcal{M}) \cup \{c_X \mid X \subseteq \mathcal{M}\}$ (a new constant c_X for every subset X of $\mathcal{M}$). Let T' be the $\mathcal{L}'$-theory consisting of all $\mathcal{L}_{\text{PA}}(\mathcal{M})$-sentences which are true in $\mathcal{M}$, together with all sentences $\{(c_X)_a = 0 \mid a \in X\}$ and all sentences $\{(c_X)_a = 1 \mid a \notin X\}$.

First, let us see that T' is consistent. Every finite subtheory of T' contains only finitely many sentences involving the constants c_X, say $(c_X)_{a_1} = \cdots = (c_X)_{a_n} = 0$ and $(c_x)_{b_1} = \cdots = (c_X)_{b_m} = 1$. Now the facts about sequence coding that we proved in PA ensure that

$$\mathcal{M} \models \exists x((x)_{a_1} = \cdots = x_{a_n} = 0 \wedge (x)_{b_1} = \cdots = (x)_{b_m} = 1)$$

and this means that every finite subtheory of T' is consistent; by Compactness, T' is consistent and has therefore a model $\mathcal{M}'$. Since $\mathcal{M}'$ satisfies all $\mathcal{L}_{\text{PA}}(\mathcal{M})$-sentences which hold in $\mathcal{M}$, $\mathcal{M}'$ is an elementary extension of $\mathcal{M}$. Furthermore, let $c_X^{\mathcal{M}'}$ be the interpretation of the constant c_X in $\mathcal{M}'$. Then for any $X \subseteq \mathcal{M}$ wwe have the set

$$Y \;=\; \{a \in \mathcal{M}' \mid \mathcal{M}' \models (c_X^{\mathcal{M}'})_a = 0\}$$

which is definable in one parameter from $\mathcal{M}'$, and satisfies $Y \cap \mathcal{M} = X$. Note that part a) now implies that $\mathcal{M}'$ is a proper extension of $\mathcal{M}$.