H10 Seminar: Homework set 17

Joep Horbach

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In both exercises: Let K be a totally real number field of degree n over $\mathbb{Q}$. Let $\sigma_1, ..., \sigma_n$ be all embeddings of K into $\mathbb{R}$. Let $a \in \mathcal{O}_K$ be such that $\sigma_1(a) \geq 2^{2n}$ and $\sigma_i(a) \leq \frac{1}{2}$ for $i = 2, 3, ..., n$.

Recall the following definitions:
We defined the sequences $x_m(a), y_m(a) \in \mathcal{O}_K$, $m \in \mathbb{N}$, by:

$$x_m(a) + y_m(a)\sqrt{a^2 - 1} = (a + \sqrt{a^2 - 1})^m$$

We defined $\epsilon = \sigma_1(a) + \sqrt{\sigma_1(a)^2 - 1}$.

Exercise 1 Prove the following facts:
(1) $\frac{\epsilon^m}{4\sigma_1(a)} < \sigma_1(y_m(a)) < \frac{\epsilon^m}{\sigma_1(a)}$
(2) $|\sigma_i(y_m(a))| < 2$ for $i = 2, 3, ..., n$
(3) $\epsilon^m/2 < \sigma_1(x_m(a)) < \epsilon^m$
(4) $|\sigma_i(x_m(a))| < 1$ for $i = 2, 3, ..., n$

Exercise 2
Let $|\sigma_i(a)| \leq \frac{1}{8}$ for all $i \neq 1$ and $m \in \mathbb{N}_{>0}$. Prove that there exists $s \in \mathbb{N}$ such that $b = x_m(a)^{2s} + a(1 - x_m(a)^2)$ satisfies the following three properties:
(i) $b \equiv 1 \mod y_m(a)$
(ii) $b \equiv a \mod x_m(a)$
(iii) $\sigma_1(b) \geq 2^{2n}$ and $\sigma_i(b) \leq \frac{1}{2}$ for $i = 2, 3, ..., n$.