Handout Seminar Presentation - Monotonicity theorem

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1 Monotonicity Theorem

Theorem 1.1 (Monotonicity). *Let $f : I \to R$ be a definable function. Then are intervals $I = I_0 \cup \cdots \cup I_k$ such that on every sub-interval I_j the function $F|I_j$ is either constant, or strictly monotone and continuous.*

Proof. Assuming the following 3 lemma's , we will derive the theorem, let I denote an interval (a, b).

Lemma 1.1. *Let $f : I \to R$ be a definable function, then there is a sub-interval of I on which F is constant or injective.*

Lemma 1.2. *Let $f : I \to R$ be a definable function, if f is injective, then f is strictly monotone on a sub-interval of I.*

Lemma 1.3. *Let $f : I \to R$ be a definable function, if f is strictly monotone, then f is continuous on a sub-interval of I.*

□

1.1 Proof of Lemma 2

$$\Phi_{++}(x) = \exists c_1, \exists c_2 \in I \Big[c_1 < x < c_2 \ \&\ \forall y \in (c_1, x) \ : \ f(y) > f(x) \\ \&\ \forall y \in (x, x_2) \ : \ f(y) > f(x) \Big],$$

$$\Phi_{--}(x) = \exists c_1, \exists c_2 \in I \Big[c_1 < x < c_2 \ \&\ \forall y \in (c_1, x) \ : \ f(y) < f(x) \\ \&\ \forall y \in (x, x_2) \ : \ f(y) < \mp f(x) \Big],$$

$$\Phi_{+-}(x) = \exists c_1, \exists c_2 \in I \Big[c_1 < x < c_2 \,\&\, \forall y \in (c_1, x) \;:\; f(y) > f(x)$$
$$\& \, \forall y \in (x, x_2) \;:\; f(y) < f(x) \Big],$$

$$\Phi_{-+}(x) = \exists c_1, \exists c_2 \in I \Big[c_1 < x < c_2 \,\&\, \forall y \in (c_1, x) \;:\; f(y) < f(x)$$
$$\& \, \forall y \in (x, x_2) \;:\; f(y) > f(x) \Big].$$

2 The Cell Decomposition Theorem

Definition 2.1. *Let $i = (i_1, i_2, \cdots i_m) \in \{0,1\}^m$. An i-cell is (definable) subset of R^m defined inductively as follows*

Base Case. *A 0-cell is a point in R, a 1-cell is an interval in R.*

Inductive Definition. . Suppose that we have already defined $(i_1, \cdots, i_m)$-cells, an $(i_1, \cdots, i_m, 0)$-cell is the graph $\Gamma(f)$ of a function f in $C_\infty(X)$ with X an $(i_1, \cdots, i_m)$-cell. An $(i_1, \cdots, i_m, 1)$-cell is a set $(f, g)_X$ where X is a $(i_1, \cdots, i_m)$-cell, $f < g \in C_\infty(X)$.

Definition 2.2. *A decomposition of R^m is a partition of R^m into finitely many cells. The definition is done by induction of the dimension m:*

(i) A decomposition of R is a collection of disjoint (0) and (1) cells such that their union is R, specifically a collection

$$\big\{(-\infty, a_1), (a_1, a_2) \cdots, (a_k, +\infty), \{a_1\}, \cdots, \{a_k\}\big\},$$

where $a_1, \cdots, a_k$ are just points in R.

(ii) A decomposition of R^{m+1} is a finite partition of R^{m+1} into cells $A_1, \cdots, A_n$ such that the set if projections $\{\pi(A_i) : 1 \le i \le n\}$ is a decomposition of R^m.

Theorem 2.1 (Cell Decomposition). *(I) Let $A_1, \cdots A_k \subset R^m$, then there is a decomposition of R^m partitioning each of $A_1, \cdots, A_k$.*

(II) For each definable function $f : A \to R, A \subset R^m$, there is a decomposition $\mathcal{D}$ of R^m such that the restriction $F|B : B \to R$ to each cell $B \in \mathcal{D}$ is continuous.

3 Finiteness Lemma

Proposition 3.1 (Finiteness Lemma)**.** *Let $A \subset R^2$ be definable and suppose that for each $x \in R$ the fiber $A_x : +\{y \in R : (x, y) \in A\}$ is finite. Then there is $N \in \mathbb{N}$ such that $|A_x| < N$ for all $x \in R$.*

$$
\begin{aligned}
\lambda(a, -) &= \lim_{\uparrow x \to a} f_{n+1}(x) \text{ if } f_{n+1} \text{ is defined on some interval}(t, a),\\
&= \infty \text{ otherwise },\\
\lambda(a, 0) &= f_{n+1}(a) \text{ if } a \in \operatorname{dom}(f_{n+1}),\\
&= \infty \text{ otherwise },\\
\lambda(a, +) &= \lim_{\downarrow x \to a} f_{n+1}(x) \text{ if } f_{n+1} \text{ is defined on some interval}(a, t),\\
&= \infty \text{ otherwise.}
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{B}_- := \{a \in \mathcal{B} \ : \ \exists y(y < \beta(a) \& (a, y) \in A)\},\\
&\mathcal{B}_+ := \{a \in \mathcal{B} \ : \ \exists y(y > \beta(a) \& (a, y) \in A)\},
\end{aligned}
$$

and the functions $\beta_- : \mathcal{B}_- \to R$ and $\beta_+ : \mathcal{B}_+ \to R$ by

$$
\begin{aligned}
&\beta_-(a) := \max\{y \ : \ y < \beta(a) \& (a, y) \in A\},\\
&\beta_+(a) := \max\{y \ : \ y > \beta(a) \& (a, y) \in A\}.
\end{aligned}
$$

Since B is infinite by assumption, one of the (definable) sets $\mathcal{B}_+ \cup \mathcal{B}_-, \mathcal{B}_+ \setminus \mathcal{B}_-, \mathcal{B}_- \setminus \mathcal{B}_+, \mathcal{B} \setminus (\mathcal{B}_- \cup \mathcal{B}_+)$ is infinite, and each of these four cases will lead to a contradiction.

Corollary 3.1. *Let $A \in R^2$ be definable such that A_x is finite for each $x \in R$. There there are points $a_1 < \cdots < a_k$ such that the intersection of A with each vertical strip $(a_i, a_{i+1}) \times R$ has the form $\Gamma(f_{i,1}) \cup \cdots \cup \Gamma(f_{i,n(i)}$ for certain definable continuous functions $f_{i,j} : (a_i, a_{i+1}) \to R$ with $f_{i,1}(x) < \cdots < f_{i,n(i)}(x)$ for each $x \in (a_i, a_{i+1})$, we have set $a_0 = -\infty$ and $a_{k+1} = +\infty$.*

The proof is this is a homework exercise.

References

[1] L. van den Dries, *Tame Topology and O-minimal Structures* (43-53) London Mathematical Society (1998) ISBN 0 521 59838 9

Homework Exercises - Due 7/11

Exercise 1(3 points) *Let $A \in R^2$ be definable such that A_x is finite for each $x \in R$. Show that there are points $a_1 < \cdots < a_k$ such that the intersection of A with each vertical strip $(a_i, a_{i+1}) \times R$ has the form $\Gamma(f_{i,1}) \cup \cdots \cup \Gamma(f_{i,n(i)}$ for certain definable continuous functions $f_{i,j} : (a_i, a_{i+1}) \to R$ with $f_{i,1}(x) < \cdots < f_{i,n(i)}(x)$ for each $x \in (a_i, a_{i+1})$, we have set $a_0 = -\infty$ and $a_{k+1} = +\infty$. (Hint: use the functions defined in the proof of the finiteless lemma and then apply the monotonicity theorem)*

We will now use the previous exercise to show that if A has infinite fibers, its boundary consists of graphs of continuous definable functions.

Exercise 2 (1 point) *Let $A \in R^n$ be definable such that A_x is infinite for each $x \in R$. Show that there are points $a_1 < \cdots < a_k$ such that the intersection of $B_{d2}(A) := \{(x, r) \in A : r \in \partial(A_x)\}$ with each vertical strip $(a_i, a_{i+1}) \times R$ has the form $\Gamma(f_{i,1}) \cup \cdots \cup \Gamma(f_{i,n(i)}$ for certain definable continuous functions $f_{i,j} : (a_i, a_{i+1}) \to R$ with $f_{i,1}(x) < \cdots < f_{i,n(i)}(x)$ for each $x \in (a_i, a_{i+1})$, we have set $a_0 = -\infty$ and $a_{k+1} = +\infty$.* (Here ∂A_x is the (topological) boundary of A_x)

Exercise 3 (2 points) *Let $f : [a, b] \to R$ be continuous and definable. Show that f takes a maximum and a minimum value on $[a, b]$.*

Exercise 4 (2 points) *Let I and J be intervals and $f : I \to R$ and $g : J \to R$ strictly monotone definable functions such that $f(I) \subset g(J)$ and $lim_{x \to r(I)} f(x) = \lim_{x \to r(J)} g(t)$ in R_∞, where $r(I)$ and $r(J)$ are the right endpoints of the intervals I and J in R_∞. Show that near these right endpoints f and g are reparametrisations of each other, that is there are subintervals I' of I and J with $r(I) = r(I'), r(J) = r(J')$ and a strictly increasing definable bijection $h : I' \to J'$ such that $f(x) = g(h(x))$ for all $s \in I'$.*